Theoretical Methods in Quantum Optics: Introduction

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Brief Overview of Quantum Optics

Quantum Optics is an exciting and dynamic field of research that encompasses a large number of topics including:

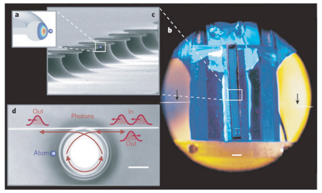
- *Laser theory and optical coherence*
- *Atomic coherence*
  - Superradiance, superfluorescence
- *Resonance fluorescence: atoms driven by laser light*
- *Generation and study of nonclassical states of light*
  - Sub-Poissonian light, antibunching, squeezing
- *Cavity quantum electrodynamics*
  - Optical bistability, single atoms and single photons
- *Laser cooling and trapping of atoms*

- *Tests of the foundations of quantum physics*
  - Schrödinger cats, Bell's inequalities, EPR paradox, decoherence, quantum measurement, quantum jumps (single atom/ion experiments)
- *Precision measurements*
  - Enhanced interferometry with nonclassical light
- *Quantum information*
  - Quantum computing, quantum communication, quantum networks

Outline of Lectures

- *Quantisation of the electromagnetic (EM) field*
  - Number states, coherent states, squeezed states
- *Quantum correlations and photon statistics*
  - Field correlation functions, optical coherence, photon correlation measurements, homodyne measurements
- *Representations of the EM field*
  - Number state-, P-, Q- and Wigner representations, optical homodyne tomography
- *Quantum phenomena in simple nonlinear optical systems*
  - Degenerate and nondegenerate parametric amplification, squeezing, nonclassical correlations, EPR paradox, teleportation
- *Master equation methods*
  - Derivation of the master equation, computation of expectation values and correlation functions, equivalent *c*-number equations, stochastic differential equations, quantum trajectories

- *Inputs and outputs in quantum optical systems*
  - Cavity modes, correlation functions, spectrum of squeezing
- *Interaction of radiation with atoms*
  - Two-state atoms, spontaneous emission, resonance fluorescence, antibunching
- *Cavity quantum electrodynamics (cavity QED)*
  - Jaynes-Cummings model, quantum collapses and revivals, cavity-enhanced spontaneous emission, transmission spectra
- *Quantum network operations in cavity QED*
  - Quantum state transfer, conditional quantum dynamics, microtoroid cavity QED



Suggested Reading

- D.F. Walls and G.J. Milburn, *Quantum Optics* (1994)
- H.J. Carmichael, *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations* (1999)
- H.J. Carmichael, *Statistical Methods in Quantum Optics 2: Non-Classical Fields* (2007)
- C.W. Gardiner and P. Zoller, *Quantum Noise, 2nd Ed.* (1999)
- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (1995)

Quantum & Atom Optics at Auckland

Theory

- Howard Carmichael, Matthew Collett, SP

Experiment (cold atoms)

- Maarten Hoogerland, Rainer Leonhardt

Theoretical Methods in Quantum Optics 1:

Quantisation of the Electromagnetic Field

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Outline

Classical electromagnetic theory is very successful in accounting for a wide variety of optical phenomena. However, there are phenomena, typically involving small photon numbers, for which the field needs to be treated quantum mechanically. In the following sections, we take up the problem of quantising the free electromagnetic field and investigate some of its properties.

Topics

- Classical Fields: Maxwell's Equations
- Field Quantisation
- Spectrum of the Energy and Number States
- Coherent States
- Quadrature Phase Operators and Phase-Space Diagrams
- Squeezed States
- Variance in the Electric Field

Classical Fields

Maxwell's equations: no sources

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r},t) &= -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \qquad & \nabla \cdot \mathbf{E}(\mathbf{r},t) = 0 \\ \nabla \times \mathbf{B}(\mathbf{r},t) &= \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) \qquad & \nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 \end{aligned}$$

Coulomb gauge: $\mathbf{B}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$ determined from vector potential $\mathbf{A}(\mathbf{r},t)$, with $\nabla \cdot \mathbf{A}(\mathbf{r},t) = 0$:

$$\begin{aligned} \mathbf{B}(\mathbf{r},t) &= \nabla \times \mathbf{A}(\mathbf{r},t) \\ \mathbf{E}(\mathbf{r},t) &= -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) \end{aligned}$$

Wave equation:

$$\nabla^2 \mathbf{A}(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{r},t)$$

Can write

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}^{(+)}(\mathbf{r},t) + \mathbf{A}^{(-)}(\mathbf{r},t), \qquad \mathbf{A}^{(-)} = (\mathbf{A}^{(+)})^*$$

Expand in discrete set of orthogonal mode functions:

$$\mathbf{A}^{(+)}(\mathbf{r},t) = \sum_k c_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t}$$

where the Fourier coefficients c_k are constant for a free field.

Mode functions $\mathbf{u}_k(\mathbf{r})$

$$\left(\nabla^2 + \frac{\omega_k^2}{c^2}\right)\mathbf{u}_k(\mathbf{r}) = 0 \qquad \nabla \cdot \mathbf{u}_k(\mathbf{r}) = 0$$

Complete orthonormal set:

$$\int_V \mathbf{u}_k^*(\mathbf{r}) \cdot \mathbf{u}_{k'}(\mathbf{r}) d\mathbf{r} = \delta_{kk'}$$

Define

$$c_k = \left(\frac{\hbar}{2\omega_k \epsilon_0}\right)^{1/2} a_k$$

so that the amplitude a_k is dimensionless. Then,

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{i} \sum_k \left(\frac{\hbar\omega_k}{2\epsilon_0}\right)^{1/2} \left[a_k \mathbf{u}_k(\mathbf{r}) e^{-\mathrm{i}\omega_k t} - a_k^* \mathbf{u}_k^*(\mathbf{r}) e^{\mathrm{i}\omega_k t}\right]$$

The Hamiltonian for the EM field is

$$\begin{aligned} H &= \frac{1}{2}\int_V \left[\epsilon_0 \mathbf{E}(\mathbf{r}, t)^2 + \frac{1}{\mu_0}\mathbf{B}(\mathbf{r}, t)^2\right] d\mathbf{r} \\ &= \frac{1}{2}\sum_k \hbar\omega_k \left(a_k^* a_k + a_k a_k^*\right) \end{aligned}$$

- Hamiltonian for an assembly of *independent harmonic oscillators*

Field Quantisation

$a_k \to \hat{a}_k$ and $a_k^* \to \hat{a}_k^\dagger$ (mutually adjoint operators).

Commutation relations

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0, \quad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}$$

Hamiltonian

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right)$$

- Dynamics of field amplitudes described by ensemble of independent *quantised harmonic oscillators*.
- State vector $|\Psi\rangle_k$ for each oscillator mode .
- State of entire field defined in tensor product space of Hilbert spaces for all modes.
- *Zero-point energy* $\hbar\omega_k/2$ (uncertainty principle).

Spectrum of the Energy and Number States

Determine from eigenvalues n_k and eigenstates $|n_k\rangle$ of operator $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$:

$$\hat{n}_k |n_k\rangle = n_k |n_k\rangle$$

Consider the state $\hat{a}_k^\dagger |n_k\rangle$. Using $[\hat{a}_k^\dagger, \hat{n}_k] = -\hat{a}_k^\dagger$ gives

$$\hat{n}_k \hat{a}_k^\dagger |n_k\rangle = \hat{a}_k^\dagger (\hat{n}_k + 1)|n_k\rangle = (n_k + 1)\hat{a}_k^\dagger |n_k\rangle$$

So, $\hat{a}_k^\dagger |n_k\rangle$ is also an eigenstate of $\hat{n}_k$, with eigenvalue $(n_k + 1)$, i.e.,

$$\hat{a}_k^\dagger |n_k\rangle = g_k |n_k + 1\rangle$$

Taking norms and using $[\hat{a}_k, \hat{a}_k^\dagger] = 1$ gives $|g_k| = \sqrt{n_k + 1}$.
Hence, up to an arbitrary phase factor

$$\hat{a}_k^\dagger |n_k\rangle = \sqrt{n_k + 1}\, |n_k + 1\rangle$$

Repeat argument $\Rightarrow$ eigenvalues $n_k, n_k + 1, n_k + 2, \ldots$ (unbounded).

Consider the state $\hat{a}_k |n_k\rangle$. Using $[\hat{a}_k, \hat{n}_k] = \hat{a}_k$ gives

$$\hat{n}_k \hat{a}_k |n_k\rangle = \hat{a}_k (\hat{n}_k - 1)|n_k\rangle = (n_k - 1)\hat{a}_k |n_k\rangle$$

So, $\hat{a}_k |n_k\rangle$ is also an eigenstate of $\hat{n}_k$, with eigenvalue $(n_k - 1)$, i.e.,

$$\hat{a}_k |n_k\rangle = d_k |n_k - 1\rangle$$

Taking norms and using $[\hat{a}_k, \hat{a}_k^\dagger] = 1$ gives $|d_k| = \sqrt{n_k}$.
Hence, up to an arbitrary phase factor

$$\hat{a}_k |n_k\rangle = \sqrt{n_k}\, |n_k - 1\rangle$$

Repeat argument $\Rightarrow$ eigenvalues $n_k, n_k - 1, n_k - 2, \ldots$.
But, sequence cannot become negative: $\langle n_k | \hat{a}_k^\dagger \hat{a}_k | n_k\rangle = n_k \geq 0$.
Lowest eigenvalue is 0 and

$$a_k |0_k\rangle = 0$$

Hence, spectrum of *number* operator $\hat{n}_k$ is the set of non-negative integers $0, 1, 2, \ldots$.

Energy eigenvalues for mode k

$$E_{n_k} = (n_k + 1/2)\hbar\omega_k \qquad (n_k = 0, 1, 2, \ldots)$$

Eigenstates: *Number* or *Fock* states

$$|n_k\rangle = \frac{(\hat{a}_k^\dagger)^{n_k}}{(n_k!)^{1/2}}|0_k\rangle \qquad (n_k = 0, 1, 2, \ldots)$$

- The Fock states are orthogonal, $\langle n_k | m_k \rangle = \delta_{mn}$, and complete,

$$\sum_{n_k=0}^{\infty} |n_k\rangle\langle n_k| = 1$$

- Form a complete set of basis vectors for a Hilbert space.

Basis vectors for entire field: tensor product over all modes

$$\prod_k |n_k\rangle$$

Photons

Discrete excitations or quanta of the EM field, corresponding to the occupation numbers $\{n_k\}$, e.g., state $|\ldots 0, 0, 1_k, 0, 0, \ldots\rangle$ described as a state with one photon in mode k.

Annihilation and creation operators

Operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ *lower and raise the photon occupation number of a state by unity*. Known as photon *annihilation operator*, and photon *creation operator*, respectively.

Notes

- Difficult to generate pure photon number states with more than a few photons.
- Most optical fields are either a superposition or mixture of number states.
- For the description of such states, alternative and more appropriate representations have been developed, e.g., the *coherent states*.

Coherent States

- Of particular importance in practical applications of the quantum theory of light.
- *Closest quantum-mechanical approach to a classical electromagnetic field of definite complex amplitude*.
- Enable a close correspondence to be made between quantum and classical correlation functions.
- Particularly appropriate for the description of fields generated by coherent sources, such as lasers and parametric oscillators.
- First discovered in connection with the quantum harmonic oscillator by Schrödinger (1926), who referred to them as states of minimum uncertainty product.
- *Relevance to quantum treatment of optical coherence and adoption in quantum optics due largely to Glauber* (1963), who coined the name 'coherent state'.

Fock representation of the coherent state

The coherent states are defined as *eigenstates of the annihilation operator*:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \qquad \left(\text{Note: } \langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|\right)$$

with α a complex number.

The Fock states form a complete set, so we can write

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$$

Substituting this form in $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ gives

$$\sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle$$

Equating coefficients of corresponding Fock states gives recursion relation

$$c_n = \frac{\alpha}{\sqrt{n}}c_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}}c_{n-2} = \ldots = \frac{\alpha^n}{\sqrt{n!}}c_0$$

So

$$|\alpha\rangle = c_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Value of $|c_0|$ determined from normalisation of the state $|\alpha\rangle$:

$$|c_0| = \exp(-|\alpha|^2/2)$$

Hence, up to an arbitrary phase factor, the coherent state is given by

$$|\alpha\rangle = \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Notes:

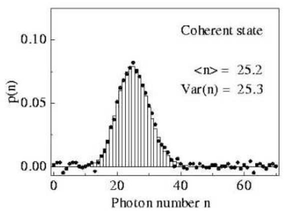
- Possible to absorb photons from a field in a coherent state repeatedly, without changing the state $\Rightarrow$ connection between the coherent state of the quantum field and a classical field
- In practice, *most measurements of the (optical) field are based on the process of photoelectric detection*, using, e.g., photomultipliers or photoconductors.
- These devices function by the *absorption of photons*; hence, the absorption operator $\hat{a}$ is the operator most closely associated with measurement of the field.
- Because the coherent states are eigenstates of the absorption operator, these states are *particularly convenient for the description of properties of the field encountered in photoelectric measurements*.

Photon number distribution

The probability $P(n)$ that n photons will be found in the state $|\alpha\rangle$ is

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{\exp(-|\alpha|^2)|\alpha|^{2n}}{n!}$$

i.e., a *Poisson distribution* in n, with mean $|\alpha|^2$.



Note:
Since the number n corresponds to the eigenvalue of the number operator $\hat{n}$, we have

$$\begin{aligned}\langle\hat{n}\rangle &= \langle\alpha|\hat{n}|\alpha\rangle = \sum_n nP(n) = |\alpha|^2 \\ \langle\hat{n}^2\rangle &= \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger[\hat{a},\hat{a}^\dagger]\hat{a}|\alpha\rangle = |\alpha|^4 + |\alpha|^2\end{aligned}$$

Coherent state as a displaced vacuum state

One can show that

$$|\alpha\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle \equiv \hat{D}(\alpha)|0\rangle$$

where $\hat{D}(\alpha)$ is the *displacement operator*.

This involves the use of the Baker-Hausdorff operator identity:

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-[\hat{A},\hat{B}]/2)$$

provided that $[\hat{A},[\hat{A},\hat{B}]] = 0 = [\hat{B},[\hat{A},\hat{B}]]$. So,

$$\begin{aligned}
\hat{D}(\alpha)|0\rangle &= \exp(-|\alpha|^2/2)\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})|0\rangle \\
&= \exp(-|\alpha|^2/2)\exp(\alpha\hat{a}^\dagger)|0\rangle \qquad (\text{since } \hat{a}|0\rangle = 0) \\
&= \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n(\hat{a}^\dagger)^n}{n!}|0\rangle \\
&= \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle
\end{aligned}$$

Properties of the displacement operator

- $\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha) = \hat{D}(-\alpha)$
- $\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a}+\alpha, \quad \hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger+\alpha^*$
- $\hat{D}^\dagger(\alpha)f(\hat{a},\hat{a}^\dagger)\hat{D}(\alpha) = f(\hat{a}+\alpha,\hat{a}^\dagger+\alpha^*)$
  for any function $f(\hat{a},\hat{a}^\dagger)$ having a power series expansion
- $\hat{D}(\alpha)\hat{D}(\beta) = \exp[(\alpha\beta^* - \alpha^*\beta)/2]\hat{D}(\alpha+\beta)$

Scalar product

The scalar product of two coherent states is

$$\langle\alpha|\beta\rangle = \exp(\alpha^*\beta - |\alpha|^2/2 - |\beta|^2/2), \quad |\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha-\beta|^2)$$

Notice that *no two coherent states are actually orthogonal to each other*, but if α and β are very different from each other, the two states are almost orthogonal.

Completeness formula

The coherent states satisfy the completeness relation

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\,d^2\alpha = 1 \qquad \left(d^2\alpha = d(\mathrm{Re}\,\alpha)d(\mathrm{Im}\,\alpha)\right)$$

so they form a basis for the representation of other states, i.e., if $|\psi\rangle$ is an arbitrary state, then

$$|\psi\rangle = \frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\psi\rangle\,d^2\alpha$$

Note:
The set of coherent states is usually said to be *over-complete*, in the sense that the states form a basis and yet are expressible in terms of each other (due to their non-orthogonality).

Time evolution

In the Schrödinger picture any state evolves in time according to

$$|\psi(t)\rangle = \exp(-\mathrm{i}\hat{H}t/\hbar)|\psi(0)\rangle$$

Consider $|\psi(0)\rangle = |\alpha\rangle$. Taking $\hat{H} = \hbar\omega(\hat{n} + 1/2)$, we have

$$\begin{aligned}|\psi(t)\rangle &= \exp(-\mathrm{i}\omega t/2)\exp(-\mathrm{i}\omega t\hat{n})|\alpha\rangle \\ &= \exp(-\mathrm{i}\omega t/2)\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp(-\mathrm{i}\omega t\hat{n})|n\rangle \\ &= \exp(-\mathrm{i}\omega t/2)\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{(\alpha e^{-\mathrm{i}\omega t})^n}{\sqrt{n!}}|n\rangle \\ &= \exp(-\mathrm{i}\omega t/2)|\alpha e^{-\mathrm{i}\omega t}\rangle\end{aligned}$$

Apart from a phase factor, this is just another coherent state of amplitude $\alpha e^{-\mathrm{i}\omega t}$. Thus the *coherent state evolves into other coherent states continuously and periodically*.

The time dependence of the expectation values of the annihilation and creation operators is given by

$$\langle\psi(t)|\hat{a}|\psi(t)\rangle = \alpha e^{-\mathrm{i}\omega t}, \qquad \langle\psi(t)|\hat{a}^\dagger|\psi(t)\rangle = \alpha^* e^{\mathrm{i}\omega t}$$

For the canonically conjugate operators $\hat{q}$ and $\hat{p}$, defined by

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a}^\dagger + \hat{a}), \qquad \hat{p} = \mathrm{i}\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a})$$

we find

$$\begin{aligned}\langle\psi(t)|\hat{q}|\psi(t)\rangle &= \sqrt{2\hbar/\omega}\,|\alpha|\cos(\omega t - \theta) \\ \langle\psi(t)|\hat{p}|\psi(t)\rangle &= -\sqrt{2\hbar\omega}\,|\alpha|\sin(\omega t - \theta)\end{aligned}$$

where we write $\alpha = |\alpha|e^{\mathrm{i}\theta}$.

This behaviour is reminiscent of a classical harmonic oscillator of frequency ω, with a well-defined complex amplitude α.

Canonical uncertainty product

The variance of $\hat{q}$ for a coherent state is

$$\langle(\Delta\hat{q}(t))^2\rangle \equiv \langle\psi(t)|(\Delta\hat{q})^2|\psi(t)\rangle \equiv \langle\psi(t)|\hat{q}^2|\psi(t)\rangle - \langle\psi(t)|\hat{q}|\psi(t)\rangle^2 = \frac{\hbar}{2\omega}$$

and that of $\hat{p}$ is $\qquad \langle(\Delta\hat{p}(t))^2\rangle = \frac{\hbar\omega}{2}$

The product of the uncertainties is then

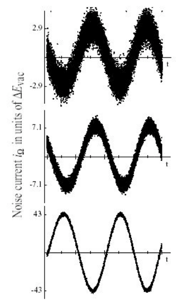
$$\langle(\Delta\hat{q}(t))^2\rangle^{1/2}\langle(\Delta\hat{p}(t))^2\rangle^{1/2} = \frac{1}{2}\hbar$$

which is the minimum allowed by quantum mechanics.

Hence, the coherent state is a *minimum uncertainty state*, behaving as nearly like a classical field as is possible.

Notes:

- The uncertainties in the canonical variables are independent of the eigenvalue α.
- Whether $\langle(\Delta\hat{q}(t))^2\rangle$ is appreciable or not compared with $\langle\hat{q}(t)\rangle^2$ depends on the magnitude $|\alpha|$.
- *The departure from classical behaviour is unimportant when $|\alpha| \gg 1$, but is significant when $|\alpha| \lesssim 1$.*



$|\alpha| \simeq 2$

$|\alpha| \simeq 5$

$|\alpha| \simeq 30$

Quadrature Phase Operators & Phase-Space Diagrams

Quadrature phase operators

The (Hermitian) quadrature phase operators, $\hat{X}_1, \hat{X}_2$, are defined by

$$\hat{a} = \frac{1}{2}(\hat{X}_1 + \mathrm{i}\hat{X}_2)$$

i.e., as the real and imaginary parts of the complex amplitude. They obey the commutation relation $[\hat{X}_1, \hat{X}_2] = 2\mathrm{i}$, with the corresponding uncertainty relation

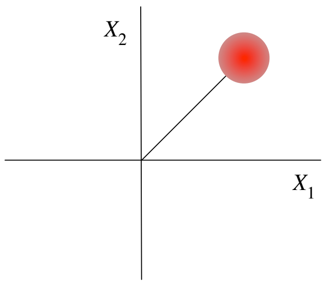
$$\langle(\Delta\hat{X}_1)^2\rangle^{1/2}\langle(\Delta\hat{X}_2)^2\rangle^{1/2} \geq 1$$

This relation *with the equals sign defines a family of minimum uncertainty states*. The coherent states are a particular example with

$$\langle(\Delta\hat{X}_1)^2\rangle = \langle(\Delta\hat{X}_2)^2\rangle = 1$$

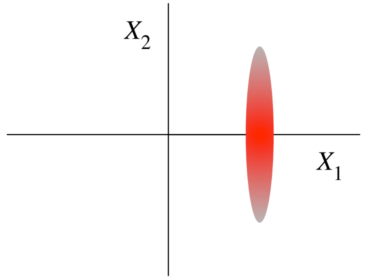
Phase-space diagrams

- A coherent state may be represented by an ‘error circle’ in a complex amplitude plane whose axes are X_1 and X_2.
- The centre of the error circle lies at $(1/2)\langle\hat{X}_1 + \mathrm{i}\hat{X}_2\rangle = \alpha$.
- The radius $\langle(\Delta\hat{X}_1)^2\rangle^{1/2} = \langle(\Delta\hat{X}_2)^2\rangle^{1/2} = 1$ accounts for the uncertainties in X_1 and X_2.



Squeezed States

- States with less uncertainty in one observable than for the vacuum state.
- Distribution of canonical variables over the phase space is distorted or “squeezed”.
- *Variance in one variable is reduced at the expense of an increase in the variance in the conjugate variable*.



Squeeze operator

The squeezed states may be generated from the vacuum by the operation of the unitary *squeeze operator*

$$\hat{S}(\epsilon) = \exp\left[\frac{1}{2}\epsilon^*\hat{a}^2 - \frac{1}{2}\epsilon(\hat{a}^\dagger)^2\right] \qquad \text{with} \quad \epsilon = re^{2\mathrm{i}\phi}$$

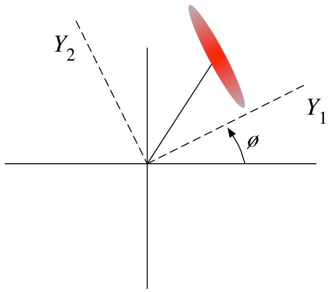
Properties of the squeeze operator:

- $\hat{S}^\dagger(\epsilon) = \hat{S}^{-1}(\epsilon) = \hat{S}(-\epsilon)$
- $\hat{S}^\dagger(\epsilon)\hat{a}\hat{S}(\epsilon) = \hat{a}\cosh(r) - \hat{a}^\dagger e^{2\mathrm{i}\phi}\sinh(r)$
- $\hat{S}^\dagger(\epsilon)(\hat{Y}_1 + \mathrm{i}\hat{Y}_2)\hat{S}(\epsilon) = \hat{Y}_1 e^{-r} + \mathrm{i}\hat{Y}_2 e^{r}$ where $\hat{Y}_1 + \mathrm{i}\hat{Y}_2 = (\hat{X}_1 + \mathrm{i}\hat{X}_2)e^{-\mathrm{i}\phi}$ is a rotated complex amplitude.
- The squeeze operator attenuates one component of the (rotated) complex amplitude and amplifies the other component. Degree of attenuation/amplification determined by $r = |\epsilon|$ = *squeeze factor*.

The squeezed state $|\alpha, \epsilon\rangle$ is obtained by first squeezing the vacuum and then displacing it:

$$|\alpha, \epsilon\rangle = \hat{D}(\alpha)\hat{S}(\epsilon)|0\rangle$$

- Expectation values and variances:
  $\langle \hat{X}_1 + i\hat{X}_2\rangle = \langle \hat{Y}_1 + i\hat{Y}_2\rangle e^{i\phi} = 2\alpha$
  $\langle(\Delta\hat{Y}_1)^2\rangle = e^{-2r}, \quad \langle(\Delta\hat{Y}_2)^2\rangle = e^{2r}$
  $\langle\hat{n}\rangle = |\alpha|^2 + \sinh^2(r)$
- The squeezed state has unequal uncertainties for Y_1 and Y_2, producing an *'error ellipse' in phase space*.
- The principal axes of the ellipse lie along the Y_1 and Y_2 axes, and the principal radii are ΔY_1 and ΔY_2.



Photon number distribution for the squeezed state $|\alpha, \epsilon\rangle$

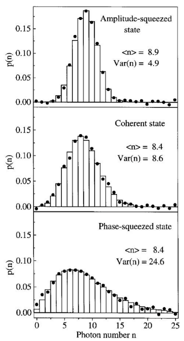
$$P(n) = (n!\mu)^{-1}\left|\frac{\nu}{2\mu}\right|^n \left|H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right)\right|^2 \exp\left(-|\beta|^2 + \frac{\nu}{2\mu}\beta^2 + \frac{\nu^*}{2\mu}\beta^{*2}\right)$$

where $H_n(x)$ are Hermite polynomials and

$$\nu = e^{2i\phi}\sinh(r), \quad \mu = \cosh(r), \quad \beta = \mu\alpha + \nu\alpha^*.$$

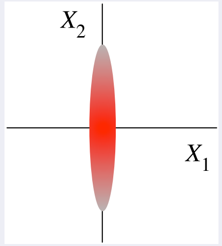
This distribution may be broader or narrower than a Poissonian distribution, depending on whether the reduced fluctuations occur in the phase (X_2) or amplitude (X_1) quadrature of the field.

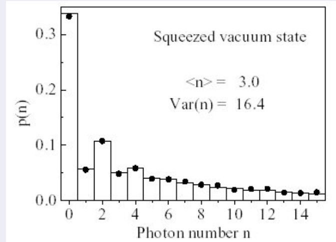
Expt: Breitenbach, Schiller, Mlynek, Nature **387**, 471 (1997)



Note

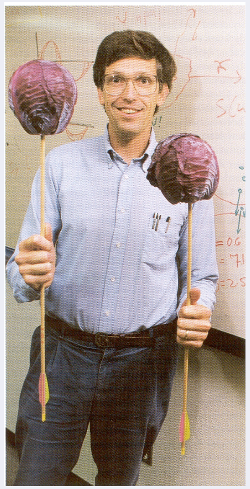
A squeezed vacuum ($\alpha = 0$) contains only *even* numbers of photons, since $H_n(0) = 0$ for n odd.

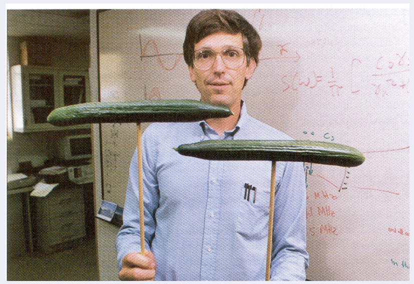




Expt: Breitenbach, Schiller, Mlynek, Nature **387**, 471 (1997)

Enhanced measurement sensitivity with squeezed states





(An experimentalist's view)

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Precision Measurement beyond the Shot-Noise Limit

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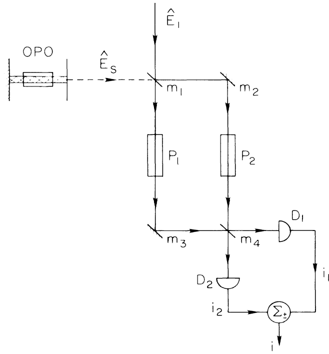
(Received 28 May 1987)

An improvement in precision beyond the limit set by the vacuum-state or zero-point fluctuations of the electromagnetic field is reported for the measurement of phase modulation in an optical interferometer. The experiment makes use of squeezed light to reduce the level of fluctuations below the shot-noise limit. An increase in the signal-to-noise ratio of 3.0 dB relative to the shot-noise limit is demonstrated, with the improvement currently limited by losses in propagation and detection and not by the degree of available squeezing.



Variance in the Electric Field

The electric field for a single mode of the EM field may be written (for a quantisation volume $\mathcal{V}$) as

$$\hat{E}(\mathbf{r},t) = \left(\frac{\hbar\omega}{2\epsilon_0\mathcal{V}}\right)^{1/2}\left[\hat{X}_1\sin(\omega t - \mathbf{k}\cdot\mathbf{r}) - \hat{X}_2\cos(\omega t - \mathbf{k}\cdot\mathbf{r})\right]$$

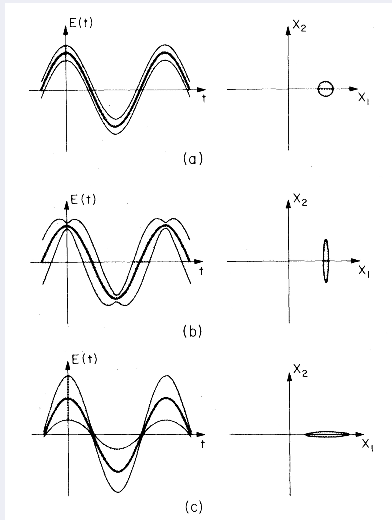
The variance $V(E) \equiv \langle(\Delta\hat{E})^2\rangle$ is

$$\begin{aligned} V(E) &= \left(\frac{2\hbar\omega}{\epsilon_0\mathcal{V}}\right)\{\, V(X_1)\sin^2(\omega t - \mathbf{k}\cdot\mathbf{r}) + V(X_2)\cos^2(\omega t - \mathbf{k}\cdot\mathbf{r}) \\ &\qquad - V(X_1,X_2)\sin[2(\omega t - \mathbf{k}\cdot\mathbf{r})]\,\} \end{aligned}$$

where $V(X_1,X_2) = \frac{1}{2}\langle\hat{X}_1\hat{X}_2 + \hat{X}_2\hat{X}_1\rangle - \langle\hat{X}_1\rangle\langle\hat{X}_2\rangle$.
For a minimum uncertainty state $V(X_1,X_2) = 0$, and hence

$$V(E) = (2\hbar\omega/\epsilon_0\mathcal{V})[\,V(X_1)\sin^2(\omega t - \mathbf{k}\cdot\mathbf{r}) + V(X_2)\cos^2(\omega t - \mathbf{k}\cdot\mathbf{r})\,]$$

- The variance of the electric field for a coherent state $[V(X_1) = V(X_2) = 1]$ is a constant with time.
- While the coherent state error circle rotates about the origin at frequency ω, it has a constant projection on the axis defining the electric field.
- *For a squeezed state, the rotation of the error ellipse leads to a variance that oscillates with frequency* 2ω.



Theoretical Methods in Quantum Optics 2:

Quantum Correlations and Photon Statistics

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Department of Physics, University of Auckland, New Zealand

29 September, 2008

Outline

We now consider correlation functions of the electromagnetic field and how they may be used in a general definition of optical coherence.

Topics

- Field-Correlation Functions
- Correlation Functions and Optical Coherence
- Photon Correlation Measurements
- Phase-Dependent Correlation Functions

Field-Correlation Functions

Experiments which detect photons ordinarily do so by absorbing them in one way or another $\Rightarrow$ *the field we measure is that associated with photon annihilation*, i.e., $\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)$.

We take the probability for the detector to absorb a photon at position $\mathbf{r}$ and time t to be proportional to

$$T_{if} = |\langle f|\hat{E}^{(+)}(\mathbf{r}, t)|i\rangle|^2$$

where $|i\rangle$ and $|f\rangle$ are the initial and final states of the field.

We consider a single vector component of the field,

$$\hat{E}^{(+)}(\mathbf{r}, t) = \tilde{\mathbf{e}}_d^* \cdot \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t), \qquad \hat{E}^{(-)}(\mathbf{r}, t) = \tilde{\mathbf{e}}_d \cdot \hat{\mathbf{E}}^{(-)}(\mathbf{r}, t)$$

with $\tilde{\mathbf{e}}_d$ a unit vector defining the particular polarisation to which the detector is sensitive.

The total count rate, or average field intensity, is obtained by summing over a complete set of final states:

$$\begin{aligned} I(\mathbf{r}, t) = \sum_f T_{if} &= \sum_f \langle i|\hat{E}^{(-)}(\mathbf{r}, t)|f\rangle \langle f|\hat{E}^{(+)}(\mathbf{r}, t)|i\rangle \\ &= \langle i|\hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t)|i\rangle \end{aligned}$$

where we have used the completeness relation $\sum_f |f\rangle\langle f| = 1$.

This result assumes a *pure* initial state $|i\rangle$. For an initial *mixed* state described by the density operator $\hat{\rho} = \sum_i P_i |i\rangle\langle i|$,

$$I(\mathbf{r}, t) = \sum_i P_i \langle i|\hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t)|i\rangle = \mathrm{Tr}\{\hat{\rho}\hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t)\}$$

- If the field is initially in the vacuum state, $\hat{\rho} = |0\rangle\langle 0|$, then

$$I(\mathbf{r}, t) = \langle 0|\hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t)|0\rangle = 0$$

The *normal ordering* of the operators (i.e., all $\hat{a}$'s to the right of all $\hat{a}^\dagger$'s) yields zero intensity for the vacuum.

- Hence, the intensity appears in terms of a *field-correlation function*.
- More generally, the correlation between the field at the space-time points $x \equiv (\mathbf{r}, t)$ and $x' \equiv (\mathbf{r}', t')$ may be written as the correlation function

$$G^{(1)}(x, x') = \mathrm{Tr}\{\rho \hat{E}^{(-)}(x)\hat{E}^{(+)}(x')\}$$

- This *first-order correlation function* of the field is sufficient to account for classical interference experiments.
- For experiments involving, e.g., *intensity correlations*, it is necessary to define higher-order correlation functions.
- The nth-order correlation function of the field is defined by

$$G^{(n)}(x_1 \ldots x_n, x_{n+1} \ldots x_{2n}) = \mathrm{Tr}\{\rho \hat{E}^{(-)}(x_1) \ldots \hat{E}^{(-)}(x_n)\hat{E}^{(+)}(x_{n+1}) \ldots \hat{E}^{(+)}(x_{2n})\}$$

Properties of the correlation functions

For any linear operator $\hat{A}$, we must have $\mathrm{Tr}\{\hat{\rho}\hat{A}^\dagger\hat{A}\} \geq 0$.

- Choosing $\hat{A} = \hat{E}^{(+)}(x)$ gives $G^{(1)}(x, x) \geq 0$
- Choosing $\hat{A} = \hat{E}^{(+)}(x_n) \ldots \hat{E}^{(+)}(x_1)$ gives
  $G^{(n)}(x_1 \ldots x_n, x_n \ldots x_1) \geq 0$
- Choosing $\hat{A} = \sum_{j=1}^n \lambda_j \hat{E}^{(+)}(x_j)$, where $\{\lambda_j\}$ is an arbitrary set of complex numbers, gives

$$\sum_{ij} \lambda_i^* \lambda_j G^{(1)}(x_i, x_j) \geq 0$$

i.e., the set of correlation functions $G^{(1)}(x_i, x_j)$ forms a matrix of coefficients for a positive definite quadratic form. Such a matrix has a positive determinant, $\det[G^{(1)}(x_i, x_j)] \geq 0$.
For $n = 2$ this gives

$$G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2) \geq \left|G^{(1)}(x_1, x_2)\right|^2$$

- For the case of two beams (1 and 2), an interesting inequality arises from the choice

$$\hat{A} = \lambda_1 \hat{E}_1^{(-)}(x)\hat{E}_1^{(+)}(x) + \lambda_2 \hat{E}_2^{(-)}(x)\hat{E}_2^{(+)}(x)$$

which gives

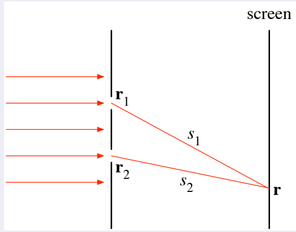
$$\left|\left\langle \hat{E}_1^{(-)}(x)\hat{E}_1^{(+)}(x)\hat{E}_2^{(-)}(x)\hat{E}_2^{(+)}(x)\right\rangle\right|^2 \leq \left\langle [\hat{E}_1^{(-)}(x)\hat{E}_1^{(+)}(x)]^2\right\rangle \left\langle [\hat{E}_2^{(-)}(x)\hat{E}_2^{(+)}(x)]^2\right\rangle$$

This proves useful in contrasting classical and quantum predictions for certain optical systems (see later).

Correlation Functions and Optical Coherence

Classical optical interference experiments correspond to a measurement of the first-order correlation function.

Young's interference experiment



The field incident on the screen at position $\mathbf{r}$ and time t is a superposition of the fields emanating from the two pin holes:

$$\hat{E}^{(+)}(\mathbf{r}, t) = u_1 \hat{E}_1^{(+)}(x_1) + u_2 \hat{E}_2^{(+)}(x_2)$$

where $x_i = (\mathbf{r}_i, t - s_i/c)$, and the coefficients $u_{1,2}$, inversely proportional to $s_{1,2}$, respectively, depend on the geometry of the experiment.

The intensity at the screen is proportional to

$$I = \text{Tr}\{\hat{\rho}\hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t)\}$$
$$= |u_1|^2 G^{(1)}(x_1,x_1) + |u_2|^2 G^{(1)}(x_2,x_2) + 2\text{Re}\{u_1^* u_2 G^{(1)}(x_1,x_2)\}$$

- First two terms = intensities from each pinhole separately.
- Third term= interference term.
- $G^{(1)}(x_1,x_2)$ in general takes on complex values. Assuming $u_2 \simeq u_1$ and absorbing these factors into the normalisation, then writing

$$G^{(1)}(x_1,x_2) = |G^{(1)}(x_1,x_2)|e^{i\Psi(x_1,x_2)}$$

gives

$$I = G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2) + 2|G^{(1)}(x_1,x_2)|\cos\{\Psi(x_1,x_2)\}$$

- Interference fringes arise from the oscillations of the cosine term. The envelope of the fringes is described by the correlation function $G^{(1)}(x_1,x_2)$.

First-order optical coherence

- The idea of coherence in optics was first associated with the possibility of producing interference fringes when two fields are superposed.
- The highest degree of optical coherence was associated with a field which exhibits fringes with maximum visibility, i.e., the larger $G^{(1)}(x_1,x_2)$ the more coherent the field.
- The magnitude of $|G^{(1)}(x_1,x_2)|$ is limited by the relation

$$|G^{(1)}(x_1,x_2)| \leq \left[G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)\right]^{1/2}$$

- The best possible fringe contrast occurs with the equality sign, so *the necessary condition for full coherence is*

$$|G^{(1)}(x_1,x_2)| = \left[G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)\right]^{1/2}$$

First-order optical coherence

It is common to use the *normalised correlation function*

$$g^{(1)}(x_1,x_2) = \frac{G^{(1)}(x_1,x_2)}{\left[G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)\right]^{1/2}}$$

in terms of which the condition for full first-order coherence becomes

$$|g^{(1)}(x_1,x_2)| = 1 \qquad \text{or} \qquad g^{(1)}(x_1,x_2) = e^{i\Psi(x_1,x_2)}$$

Visibility

The visibility of the fringes is given by

$$v = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \equiv |g^{(1)}(x_1,x_2)|\frac{2(I_1 I_2)^{1/2}}{I_1 + I_2}$$

with $I_i = G^{(1)}(x_i,x_i)$.

- If the fields incident on the pinholes have equal intensities, the fringe visibility is simply equal to $|g^{(1)}|$.
- Hence, the *condition for first-order optical coherence* $|g^{(1)}| = 1$ *corresponds to the condition of maximum fringe visibility*.

General definition of first-order coherence

A more general definition of first-order coherence of the field is that the *first-order correlation function factorises*:

$$G^{(1)}(x_1, x_2) = \varepsilon^{(-)}(x_1)\varepsilon^{(+)}(x_2)$$

For a field in an eigenstate of the operator $\hat{E}^{(+)}$ this factorisation holds; *coherent states* are an example of such a field.

General definition of nth-order coherence

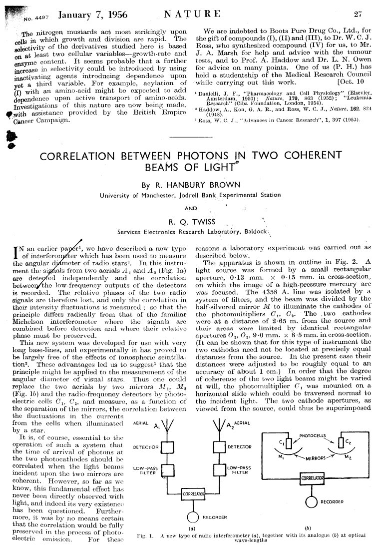
Similarly, the condition for nth-order optical coherence is that the *nth-order correlation function factorises*:

$$G^{(n)}(x_1 \dots x_n, x_{n+1} \dots x_{2n}) = \varepsilon^{(-)}(x_1) \dots \varepsilon^{(-)}(x_n)\varepsilon^{(+)}(x_{n+1}) \dots \varepsilon^{(+)}(x_{2n})$$

Again, the *coherent states* possess nth-order optical coherence.

Photon Correlation Measurements

- The first experiment performed outside the domain of one photon optics was the *intensity correlation experiment of Hanbury-Brown and Twiss (1956)*.
- Although the original experiment involved the analogue correlation of photocurrents, later versions used photon counters and digital correlations and were truly photon correlation measurements.



No. 4497 January 7, 1956 NATURE 27

The nitrogen mustards act most strikingly upon cells in which growth and division are rapid. The selectivity of the derivatives studied here is based on at least two cellular variables—growth-rate and enzyme content. It seems probable that a further increase in selectivity could be introduced by using inactivating agents introducing dependence upon yet a third variable. For example, acylation of (I) with an amino-acid might be expected to add dependence upon active transport of amino-acids. Investigations of this nature are now being made, with assistance provided by the British Empire Cancer Campaign.

We are indebted to Boots Pure Drug Co., Ltd., for the gift of compounds (I), (II) and (III), to Dr. W. C. J. Ross, who synthesized compound (IV) for us, to Mr. J. A. Marsh for help and advice with the tumour tests, and to Prof. A. Haddow and Dr. L. N. Owen for advice on many points. One of us (P. H.) has held a studentship of the Medical Research Council while carrying out this work. [Oct. 10

[1] Danielli, J. F., "Pharmacology and Cell Physiology" (Elsevier, Amsterdam, 1950); *Nature*, **170**, 863 (1952); "Leukemia Research" (Ciba Foundation, London, 1954).
[2] Haddow, A., Kon, G. A. R., and Ross, W. C. J., *Nature*, **162**, 824 (1948).
[3] Ross, W. C. J., "Advances in Cancer Research", **1**, 397 (1953).

CORRELATION BETWEEN PHOTONS IN TWO COHERENT BEAMS OF LIGHT

By R. HANBURY BROWN
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AND
R. Q. TWISS
Services Electronics Research Laboratory, Baldock

IN an earlier paper[1], we have described a new type of interferometer which has been used to measure the angular diameter of radio stars[2]. In this instrument the signals from two aerials A_1 and A_2 (Fig. 1*a*) are detected independently and the correlation between the low-frequency outputs of the detectors is recorded. The relative phases of the two radio signals are therefore lost, and only the correlation in their intensity fluctuations is measured; so that the principle differs radically from that of the familiar Michelson interferometer where the signals are combined before detection and where their relative phase must be preserved.

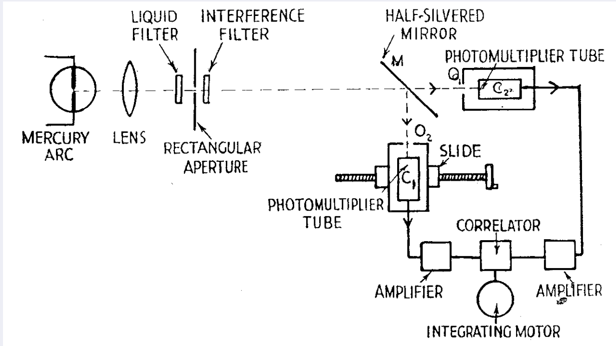
This new system was developed for use with very long base-lines, and experimentally it has proved to be largely free of the effects of ionospheric scintillation[2]. These advantages led us to suggest[1] that the principle might be applied to the measurement of the angular diameter of visual stars. Thus one could replace the two aerials by two mirrors M_1, M_2 (Fig. 1*b*) and the radio-frequency detectors by photoelectric cells C_1, C_2, and measure, as a function of the separation of the mirrors, the correlation between the fluctuations in the currents from the cells when illuminated by a star.

It is, of course, essential to the operation of such a system that the time of arrival of photons at the two photocathodes should be correlated when the light beams incident upon the two mirrors are coherent. However, so far as we know, this fundamental effect has never been directly observed with light, and indeed its very existence has been questioned. Furthermore, it was by no means certain that the correlation would be fully preserved in the process of photoelectric emission. For these reasons a laboratory experiment was carried out as described below.

The apparatus is shown in outline in Fig. 2. A light source was formed by a small rectangular aperture, 0·13 mm. × 0·15 mm. in cross-section, on which the image of a high-pressure mercury arc was focused. The 4358 A. line was isolated by a system of filters, and the beam was divided by the half-silvered mirror M to illuminate the cathodes of the photomultipliers C_1, C_2. The two cathodes were at a distance of 2·65 m. from the source and their areas were limited by identical rectangular apertures O_1, O_2, 9·0 mm. × 8·5 mm. in cross-section. (It can be shown that for this type of instrument the two cathodes need not be located at precisely equal distances from the source. In the present case their distances were adjusted to be roughly equal to an accuracy of about 1 cm.) In order that the degree of coherence of the two light beams might be varied at will, the photomultiplier C_1 was mounted on a horizontal slide which could be traversed normal to the incident light. The two cathode apertures, as viewed from the source, could thus be superimposed

Fig. 1. A new type of radio interferometer (*a*), together with its analogue (*b*) at optical wave-lengths

Experimental setup of Hanbury-Brown and Twiss



- In essence, these experiments measure the *joint probability of detecting a photon at time t and another at time $t + \tau$*.
- This may be written as an intensity or photon-number correlation function, i.e., the measured quantity is the *normally-ordered correlation function*

$$\begin{aligned} G^{(2)}(\tau) &= \langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t)\rangle \\ &= \langle : \hat{I}(t)\hat{I}(t+\tau) : \rangle \;\propto\; \langle : \hat{n}(t)\hat{n}(t+\tau) : \rangle \end{aligned}$$

Note that we assume a *stationary* field, i.e., $G^{(2)}(t, \tau) = G^{(2)}(\tau)$.

Normalised second-order correlation function

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2}$$

- For a field that possesses second-order coherence

$$G^{(2)}(\tau) = \varepsilon^{(-)}(t)\varepsilon^{(-)}(t+\tau)\varepsilon^{(+)}(t+\tau)\varepsilon^{(+)}(t) = [G^{(1)}(0)]^2$$

and $g^{(2)}(\tau) = 1$.

Classical fields

For a fluctuating classical (single mode) field we may introduce a probability distribution $P(\varepsilon)$ describing the probability of the field $E^{(+)}(\varepsilon, t)$ having the amplitude ε, where

$$E^{(+)}(\varepsilon, t) = \mathrm{i}\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \varepsilon\, \mathrm{e}^{-\mathrm{i}\omega t}$$

For zero time delay, $\tau = 0$, we may write for this single-mode field

$$g^{(2)}(0) = 1 + \frac{\int P(\varepsilon)(|\varepsilon|^2 - \langle|\varepsilon|^2\rangle)^2\, \mathrm{d}^2\varepsilon}{(\langle|\varepsilon|^2\rangle)^2}$$

An important point to note is that *for classical fields the probability distribution $P(\varepsilon)$ is positive*, and hence one must have $g^{(2)}(0) \geq 1$.

Hence, for a field with a Lorentzian spectrum

$$g^{(2)}(\tau) = 1 + \mathrm{e}^{-\gamma\tau}$$

and for a field with a Gaussian spectrum

$$g^{(2)}(\tau) = 1 + \mathrm{e}^{-\gamma^2\tau^2}$$

where γ is the *spectral linewidth*.

- For $\tau \gg \tau_c = \gamma^{-1}$ (the correlation time of the light), the correlation function factorises and $g^{(2)}(\tau) \to 1$.
- The increased value of $g^{(2)}(\tau)$ for $\tau < \tau_c$ for chaotic light over coherent light $[g^{(2)}(0)_{\text{chaotic}} = 2g^{(2)}(0)_{\text{coherent}}]$ is due to the increased intensity fluctuations in the chaotic light field.
- *There is a high probability that the photon that triggers the counter arrives during a high intensity fluctuation, hence there is a high probability that a second photon will be detected arbitrarily soon.*

Field with Gaussian statistics

For a stationary field obeying Gaussian statistics, with zero mean amplitude, $\langle E^{(-)}(\varepsilon, t)\rangle = 0$ (i.e., a *chaotic field*),

$$\begin{aligned}
&\langle E^{(-)}(\varepsilon,t)E^{(-)}(\varepsilon,t+\tau)E^{(+)}(\varepsilon,t+\tau)E^{(+)}(\varepsilon,t)\rangle \\
&\quad= \langle E^{(-)}(\varepsilon,t)E^{(-)}(\varepsilon,t+\tau)\rangle\, \langle E^{(+)}(\varepsilon,t+\tau)E^{(+)}(\varepsilon,t)\rangle \\
&\quad\quad+ \langle E^{(-)}(\varepsilon,t)E^{(+)}(\varepsilon,t)\rangle\, \langle E^{(-)}(\varepsilon,t+\tau)E^{(+)}(\varepsilon,t+\tau)\rangle \\
&\quad\quad+ \langle E^{(-)}(\varepsilon,t)E^{(+)}(\varepsilon,t+\tau)\rangle\, \langle E^{(-)}(\varepsilon,t+\tau)E^{(+)}(\varepsilon,t)\rangle
\end{aligned}$$

For fields with no phase-dependent fluctuations the first term is zero. Then,

$$G^{(2)}(\tau) = G^{(1)}(0)^2 + \left|G^{(1)}(\tau)\right|^2 \quad \text{or} \quad g^{(2)}(\tau) = 1 + \left|g^{(1)}(\tau)\right|^2$$

Now, $G^{(1)}(\tau)$ is the Fourier transform of the *spectrum* of the field:

$$S(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau} G^{(1)}(\tau)$$

Photon bunching

- This effect is called *photon bunching* and was first detected by Hanbury-Brown and Twiss.
- Later experiments showed excellent agreement with the theoretical predictions.

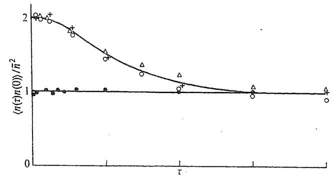


FIG. 6.3. Measured correlations between pairs of photon counts as functions of their time separation τ for laser light (filled circles) and chaotic light of Gaussian frequency distribution (all other points). The theoretical curves are the same as in Fig. 3.12. (After F. T. Arecchi *et al.*, ref. 9).

- Note, however, that the above analysis *does not rely on any quantisation of the field*, but may be deduced from a purely classical analysis with a fluctuating field amplitude.

Quantum mechanical fields

We now consider some *single-mode quantum-mechanical fields*, for which

$$g^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle^2} = 1 + \frac{V(n) - \bar{n}}{\bar{n}^2}$$

with $V(n) = \langle (\hat{a}^\dagger \hat{a})^2\rangle - \langle \hat{a}^\dagger \hat{a}\rangle^2$.

- *Coherent state*: For a coherent state $|\alpha\rangle$, $V(n) = \bar{n}$ and

$$g^{(2)}(0) = 1$$

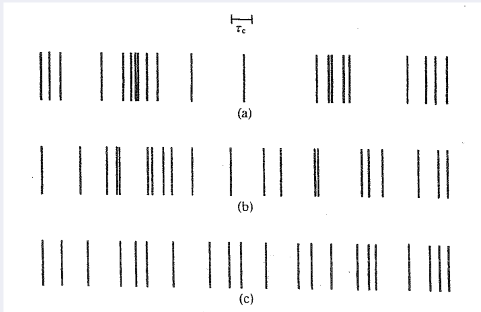
- *Number state*: For a number state $|n\rangle$, $V(n) = 0$ and

$$g^{(2)}(0) = 1 - \frac{1}{n}, \quad n > 1$$

Photon antibunching

- If $g^{(2)}(\tau) < g^{(2)}(0)$, there is a tendency for photons to arrive in pairs. This situation is referred to as *photon bunching*.
- The converse situation, $g^{(2)}(\tau) > g^{(2)}(0)$, is called *photon antibunching*.
- Noting that $g^{(2)}(\tau) \to 1$ for sufficiently large τ, a field with $g^{(2)}(0) < 1$ will always exhibit antibunching on some time scale.
- A value of $g^{(2)}(0)$ less than unity could not have been predicted by a classical analysis, i.e., *photon antibunching is a feature peculiar to the quantum mechanical nature of the EM field*.

Comparison of photon counting sequences



$g^{(2)}(0) > 1$

$g^{(2)}(0) = 1$

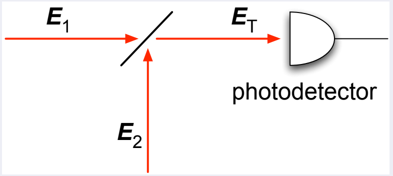
$g^{(2)}(0) < 1$

Phase-Dependent Correlation Functions

- The "even-ordered" correlation functions, such as the second-order correlation function $G^{(2)}$, contain no phase information and are a measure of the fluctuations in the photon number.
- The "odd-ordered" correlation functions $G^{(n,m)}(x_1 \dots x_n, x_{n+1} \dots x_{n+m})$ with $n \neq m$ contain information about the phase fluctuations of the field. For example, the variances in the quadrature phases, $V(X_1)$, $V(X_2)$, depend on these functions.

Homodyne measurements

- The usual scheme for making quadrature phase measurements involves mixing (or *homodyning*) the signal field (E_1) with a reference signal (E_2), known as the *local oscillator*, before photodetection.



- Homodyning with a reference signal of fixed phase gives the phase sensitivity necessary to yield the quadrature variances.

Consider two single-mode fields of the same frequency ω:

$$\begin{aligned} E_1(\mathbf{r},t) &= \mathrm{i}\left(\frac{\hbar\omega}{2V\epsilon_0}\right)^{1/2}\left[\hat{a}e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{a}^\dagger e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \\ E_2(\mathbf{r},t) &= \mathrm{i}\left(\frac{\hbar\omega}{2V\epsilon_0}\right)^{1/2}\left[\hat{b}e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{b}^\dagger e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \end{aligned}$$

combined on a beamsplitter with transmittivity η.

The total field incident on the photodetector after combination is

$$E_T(\mathbf{r},t) = \mathrm{i}\left(\frac{\hbar\omega}{2V\epsilon_0}\right)^{1/2}\left[\hat{c}\,e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{c}^\dagger e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

where $\hat{c} = \sqrt{\eta}\,\hat{a} + \mathrm{i}\sqrt{1-\eta}\,\hat{b}$.

Note: We have included a $\pi/2$ phase shift between the reflected and transmitted beams at the beamsplitter.

The photodetector responds to the moments of $\hat{c}^\dagger\hat{c}$, so the mean photocurrent in the detector is proportional to

$$\langle\hat{c}^\dagger\hat{c}\rangle = \eta\langle\hat{a}^\dagger\hat{a}\rangle + (1-\eta)\langle\hat{b}^\dagger\hat{b}\rangle - \mathrm{i}\sqrt{\eta(1-\eta)}\left(\langle\hat{a}\rangle\langle\hat{b}^\dagger\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{b}\rangle\right)$$

We take the field $\hat{E}_2$ to be the local oscillator and assume it to be in a *coherent state of large amplitude* β (so we may neglect the term $\eta\langle\hat{a}^\dagger\hat{a}\rangle$). Then

$$\langle\hat{c}^\dagger\hat{c}\rangle \simeq (1-\eta)|\beta|^2 + |\beta|\sqrt{\eta(1-\eta)}\,\langle\hat{X}_{\theta+\pi/2}\rangle$$

where $\hat{X}_\theta \equiv \hat{a}e^{-\mathrm{i}\theta} + \hat{a}^\dagger e^{\mathrm{i}\theta}$, and θ is the phase of β.

- When the contribution from the reflected local oscillator intensity is subtracted, the mean photocurrent is proportional to the mean quadrature phase amplitude of the signal field defined with respect to the local oscillator phase.

- Fluctuations in the photocurrent will be determined by the variance of $\hat{n}_c \equiv \hat{c}^\dagger\hat{c}$.
- For an intense local oscillator in a coherent state, this is

$$V(n_c) \simeq (1-\eta)^2|\beta|^2 + |\beta|^2\eta(1-\eta)V(X_{\theta+\pi/2})$$

- So, the signal-field quadrature variances, which depend on "odd-order" correlation functions, can also be determined from the photocurrent.

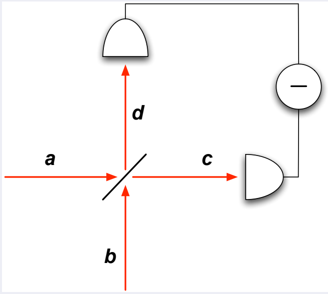
Balanced homodyne detection

- In *balanced homodyne detection*, the outputs of a 50:50 beamsplitter are directed to photodetectors and the difference between the measured photocurrents is taken.
- The difference current is proportional to

$$\langle \hat{c}^\dagger \hat{c} - \hat{d}^\dagger \hat{d} \rangle = |\beta| \langle \hat{X}_{\theta+\pi/2} \rangle$$

and the variance

$$V(\hat{c}^\dagger \hat{c} - \hat{d}^\dagger \hat{d}) = |\beta|^2 V(\hat{X}_{\theta+\pi/2})$$



Theoretical Methods in Quantum Optics 3:
Representations of the Electromagnetic Field

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29 September, 2008

Outline

For a full quantum statistical treatment of the electromagnetic field, the description of the system is best carried out in terms of the density operator $\hat{\rho}$. We now consider a number of possible representations for the density operator.

Topics

- Number State Representation
- Glauber-Sudarshan P-Representation
- Q Representation
- Wigner Representation
- Optical Homodyne Tomography

Number State Representation

The number states form a complete set and hence we can write

$$\hat{\rho} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} |n\rangle\langle m|$$

- The expansion coefficients c_{nm} are complex and there are an infinite number of them.
- Hence, the general expansion is often not very useful, particularly for problems where the phase-dependent properties of the EM field are important (and hence the full expansion is necessary).

- However, in certain cases where only the photon number distribution is of interest the reduced expansion

$$\hat{\rho} = \sum_{n=0}^{\infty} P(n) |n\rangle\langle n|$$

may be used. This is not a general representation for all fields, but may prove useful for certain fields; for example, a *chaotic field*, which has no phase information, and for which

$$P(n) = \frac{1}{1+\overline{n}} \left(\frac{\overline{n}}{1+\overline{n}} \right)^n$$

where $\overline{n}$ is the mean number of photons.

Glauber-Sudarshan P Representation

The Glauber-Sudarshan P representation relies on the fact that the coherent states are not orthogonal, forming an overcomplete basis.

As a consequence, it is often possible to expand $\hat{\rho}$ as a diagonal sum over coherent states:

$$\hat{\rho} = \int \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| P(\alpha)$$

where $\mathrm{d}^2\alpha \equiv \mathrm{d}(\mathrm{Re}\{\alpha\})\mathrm{d}(\mathrm{Im}\{\alpha\})$.

This representation for $\hat{\rho}$ is appealing because the function $P(\alpha)$ plays a role which is rather analogous to a classical probability distribution.

Expectation values of operators written in normal order are given by

$$\begin{aligned}\langle \hat{a}^{\dagger p}\hat{a}^{q}\rangle = \mathrm{Tr}\left[\hat{\rho}\hat{a}^{\dagger p}\hat{a}^{q}\right] &= \mathrm{Tr}\left[\int \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| P(\alpha)\hat{a}^{\dagger p}\hat{a}^{q}\right] \\ &= \int \mathrm{d}^2\alpha \, P(\alpha)\, \alpha^{*p}\alpha^{q}\end{aligned}$$

Normally-ordered averages are therefore calculated in the same way that averages are calculated in classical statistics, with $P(\alpha)$ playing the role of the probability distribution.

Setting $p = q = 0$ gives

$$\int \mathrm{d}^2\alpha \, P(\alpha) = 1$$

so $P(\alpha)$ is also normalised like a classical probability distribution.

The 2nd-order correlation function may be expressed as

$$g^{(2)}(0) = 1 + \frac{\int \mathrm{d}^2\alpha \, P(\alpha) \, \left[|\alpha|^2 - \langle|\alpha|^2\rangle\right]^2}{\left[\int \mathrm{d}^2\alpha \, P(\alpha)\, |\alpha|^2\right]^2}$$

which looks functionally identical to the expression for classical fields.

Similarly, for the quadrature variances we find

$$\begin{aligned}\langle(\Delta\hat{X}_1)^2\rangle &= 1 + \int \mathrm{d}^2\alpha \, P(\alpha) \left[(\alpha + \alpha^*) - (\langle\alpha\rangle + \langle\alpha^*\rangle)\right]^2 \\ \langle(\Delta\hat{X}_2)^2\rangle &= 1 + \int \mathrm{d}^2\alpha \, P(\alpha) \left[\left(\frac{\alpha - \alpha^*}{i}\right) - \left(\frac{\langle\alpha\rangle - \langle\alpha^*\rangle}{i}\right)\right]^2\end{aligned}$$

The condition for antibunching, $g^{(2)}(0) < 1$, and the condition for squeezing, $\langle(\Delta\hat{X}_k)^2\rangle < 1$, *evidently require that $P(\alpha)$ takes on negative values in some regions of the complex plane*.

Notes:

- The nonorthogonality of the coherent states gives

$$\langle\alpha|\hat{\rho}|\alpha\rangle = \int \mathrm{d}^2\beta \, e^{-|\beta-\alpha|^2} P(\beta)$$

  where we have used $|\langle\alpha|\beta\rangle|^2 = \exp(-|\beta - \alpha|^2)$.
- Hence, $\langle\alpha|\hat{\rho}|\alpha\rangle \neq P(\alpha)$; only when $P(\beta)$ is sufficiently broad compared to the Gaussian 'filter' does it approximate a probability.
- Also, although the probability $\langle\alpha|\hat{\rho}|\alpha\rangle$ must be positive, $P(\alpha)$ is not required to be so. Thus, unlike a classical probability, $P(\alpha)$ can take negative values over a limited range.
- Hence, $P(\alpha)$ is often referred to as a *quasidistribution function*.

Can we find a P representation for any density operator?

Consider

$$\begin{aligned}\mathrm{Tr}\left(\hat\rho\, e^{iz^*\hat a^\dagger} e^{iz\hat a}\right) &= \mathrm{Tr}\left\{\left[\int d^2\alpha\, |\alpha\rangle\langle\alpha|\, P(\alpha)\right] e^{iz^*\hat a^\dagger} e^{iz\hat a}\right\} \\ &= \int d^2\alpha\, P(\alpha) e^{iz^*\alpha^*} e^{iz\alpha}\end{aligned}$$

This is just a 2-D Fourier transform. The inverse transform gives

$$P(\alpha) = \frac{1}{\pi^2}\int d^2z\, \mathrm{Tr}\left(\hat\rho\, e^{iz^*\hat a^\dagger} e^{iz\hat a}\right) e^{-iz^*\alpha^*} e^{-iz\alpha}$$

If the Fourier transform of the function defined by the trace exists for a given density operator $\hat\rho$, we have our P distribution representing that density operator.

Coherent state $\hat\rho = |\alpha_0\rangle\langle\alpha_0|$

$$\begin{aligned}P(\alpha) &= \frac{1}{\pi^2}\int d^2z\, e^{-iz^*(\alpha^*-\alpha_0^*)} e^{-iz(\alpha-\alpha_0)} \\ &= \delta^{(2)}(\alpha-\alpha_0) \equiv \delta(x-x_0)\,\delta(y-y_0)\end{aligned}$$

where $\alpha = x + iy$ and $\alpha_0 = x_0 + iy_0$.

Chaotic (thermal) state $\hat\rho = \sum_n P(n)|n\rangle\langle n|$

$$P(\alpha) = \frac{1}{\pi\bar n}\exp\left(-\frac{|\alpha|^2}{\bar n}\right)$$

where $\bar n$ is the mean photon number.

Number state $\hat\rho = |l\rangle\langle l|$

$$P(\alpha) = \frac{1}{\pi^2}\int d^2z\left[\sum_{k=0}^{l}\frac{(-1)^k|z|^{2k}}{k!}\frac{l!}{k!(l-k)!}\right] e^{-iz^*\alpha^*} e^{-iz\alpha}$$

Noting that

$$\delta^{(2)}(\alpha) = \frac{1}{\pi^2}\int d^2z\, e^{-iz^*\alpha^*} e^{-iz\alpha}$$

and using the ordinary rules of differentiation inside the integral, we may write

$$P(\alpha) = \sum_{k=0}^{l}\frac{l!}{k!(l-k)!}\frac{1}{k!}\frac{\partial^{2k}}{\partial\alpha^k\partial\alpha^{*k}}\delta^{(2)}(\alpha)$$

This (generalised) function is much more singular than any classical probability distribution $\Longleftrightarrow$ *the number state $|l\rangle$ is a quantum state of the field having no classical counterpart*.

Quantum characteristic functions

The *normally ordered quantum characteristic function* is defined by

$$\chi_N(z, z^*) = \mathrm{Tr}\left(\hat\rho\, e^{iz^*\hat a^\dagger} e^{iz\hat a}\right)$$

Analogous to a classical characteristic function, one may write for the normally-ordered moments:

$$\langle\hat a^{\dagger p}\hat a^q\rangle = \mathrm{Tr}\left(\hat\rho\,\hat a^{\dagger p}\hat a^q\right) = \left.\frac{\partial^{p+q}}{\partial(iz^*)^p\partial(iz)^q}\chi_N(z, z^*)\right|_{z=z^*=0}$$

- We have

$$P(\alpha, \alpha^*) = \frac{1}{\pi^2}\int d^2z\, \chi_N(z, z^*)\, e^{-iz^*\alpha^*} e^{-iz\alpha}$$

We may also define the *antinormally ordered characteristic function*

$$\chi_A(z,z^*) = \mathrm{Tr}\left(\hat{\rho}\, e^{iz\hat{a}} e^{iz^*\hat{a}^\dagger}\right)$$

and the *symmetrically ordered characteristic function*

$$\chi_S(z,z^*) = \mathrm{Tr}\left(\hat{\rho}\, e^{iz^*\hat{a}^\dagger + iz\hat{a}}\right)$$

Q Representation

The distribution $Q(\alpha,\alpha^*)$ is defined as the Fourier transform of the antinormally ordered characteristic function $\chi_A(z,z^*)$:

$$Q(\alpha,\alpha^*) = \pi^{-2}\int d^2z\, \chi_A(z,z^*)\, e^{-iz^*\alpha^*} e^{-iz\alpha}$$

In contrast to the P distribution, which gives the normally ordered moments, the Q distribution gives the *antinormally ordered moments*:

$$\langle \hat{a}^q \hat{a}^{\dagger p}\rangle = \int d^2\alpha\, Q(\alpha,\alpha^*)\, \alpha^{*p}\alpha^q$$

The Q representation has a simple relationship to the coherent states:

$$\begin{aligned} Q(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int d^2z\, \mathrm{Tr}\left[\hat{\rho}\, e^{iz\hat{a}}\left(\frac{1}{\pi}\int d^2\lambda\, |\lambda\rangle\langle\lambda|\right) e^{iz^*\hat{a}^\dagger}\right] e^{-iz^*\alpha^*} e^{-iz\alpha} \\ &= \frac{1}{\pi}\int d^2\lambda\, \langle\lambda|\hat{\rho}|\lambda\rangle \left[\frac{1}{\pi^2}\int d^2z\, e^{iz^*(\lambda^*-\alpha^*)} e^{iz(\lambda-\alpha)}\right] \\ &= \frac{1}{\pi}\int d^2\lambda\, \langle\lambda|\hat{\rho}|\lambda\rangle\, \delta^{(2)}(\lambda-\alpha) \\ &= \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \geq 0 \end{aligned}$$

Thus, $\pi Q(\alpha,\alpha^*)$ is strictly a probability – *the probability for observing the coherent state* $|\alpha\rangle$.

Relationship between $Q(\alpha,\alpha^*)$ and $P(\alpha,\alpha^*)$

$$\begin{aligned} Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle &= \frac{1}{\pi}\int d^2\beta\, P(\beta,\beta^*)\, |\langle\alpha|\beta\rangle|^2 \\ &= \frac{1}{\pi}\int d^2\beta\, P(\beta,\beta^*)\, e^{-|\alpha-\beta|^2} \end{aligned}$$

So, the Q function is a Gaussian convolution of the P function, which accounts for its more well-behaved properties.

Examples:

Coherent state $|\beta\rangle$

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}|\langle\alpha|\beta\rangle|^2 = \frac{1}{\pi}\,\mathrm{e}^{-|\alpha-\beta|^2}$$

Number state $|n\rangle$

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}|\langle\alpha|n\rangle|^2 = \frac{1}{\pi}\frac{|\alpha|^{2n}\mathrm{e}^{-|\alpha|^2}}{n!}$$

Wigner Representation

The Wigner distribution $W(\alpha,\alpha^*)$ is the Fourier transform of the symmetrically ordered characteristic function $\chi_S(z,z^*)$:

$$W(\alpha,\alpha^*) = \pi^{-2}\int d^2z\,\chi_S(z,z^*)\,\mathrm{e}^{-iz^*\alpha^*}\mathrm{e}^{-iz\alpha}$$

The moments of $W(\alpha,\alpha^*)$ are equal to the averages of *symmetrically ordered products* of creation and annihilation operators:

$$\langle(\hat{a}^{\dagger p}\hat{a}^q)_S\rangle = \int d^2\alpha\,W(\alpha,\alpha^*)\,\alpha^{*p}\alpha^q$$

where $(\hat{a}^{\dagger p}\hat{a}^q)_S$ denotes the average of $(p+q)!/(p!q!)$ possible orderings of p creation operators and q annihilation operators.
For example,

$$(\hat{a}^\dagger\hat{a})_S = \frac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger),\quad (\hat{a}^{\dagger 2}\hat{a})_S = \frac{1}{3}(\hat{a}^{\dagger 2}\hat{a}+\hat{a}^\dagger\hat{a}\hat{a}^\dagger+\hat{a}\hat{a}^{\dagger 2}),\quad \ldots$$

Relationship between $W(\alpha,\alpha^*)$ and $P(\alpha,\alpha^*)$

Noting that $\chi_S(z,z^*) = \chi_N(z,z^*)\exp(-|z|^2/2)$ (Baker-Hausdorff theorem), we can write

$$\begin{aligned}
W(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int d^2z\,\chi_N(z,z^*)\,\mathrm{e}^{-|z|^2/2}\mathrm{e}^{-iz^*\alpha^*}\mathrm{e}^{-iz\alpha} \\
&= \frac{1}{\pi^2}\int d^2z\int d^2\beta\,P(\beta,\beta^*)\,\mathrm{e}^{iz^*\beta^*}\mathrm{e}^{iz\beta}\,\mathrm{e}^{-|z|^2/2}\mathrm{e}^{-iz^*\alpha^*}\mathrm{e}^{-iz\alpha} \\
&= \frac{1}{\pi^2}\int d^2\beta\,P(\beta,\beta^*)\int d^2z\,\mathrm{e}^{-|z|^2/2+iz^*(\beta^*-\alpha^*)+iz(\beta-\alpha)} \\
&= \frac{2}{\pi}\int d^2\beta\,P(\beta,\beta^*)\,\mathrm{e}^{-2|\beta-\alpha|^2}
\end{aligned}$$

So, the Wigner function is also a Gaussian convolution of the P function, although the Gaussian is narrower than for the Q function.

Coherent state $|\alpha_0\rangle = |(1/2)[x_1^{(0)} + ix_2^{(0)}]\rangle$

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\exp(-2|\alpha-\alpha_0|^2)$$

or, in terms of quadrature variables,

$$W(x_1,x_2) = \frac{2}{\pi}\exp\left[-\frac{1}{2}(x_1-x_1^{(0)})^2 - \frac{1}{2}(x_2-x_2^{(0)})^2\right]$$

The *contour* of the Wigner function can be defined by

$$(x_1-x_1^{(0)})^2 + (x_2-x_2^{(0)})^2 = 1$$

which we identify with the *error area* introduced earlier in the context of quadrature phase diagrams, i.e., the error area for the coherent state $|\alpha_0\rangle$ is a circle with radius one centred on the point $(x_1^{(0)}, x_2^{(0)})$.

Squeezed state $|\alpha_0, r\rangle$

$$W(x_1, x_2) = \frac{2}{\pi} \exp\left[-\frac{1}{2}(x_1 - x_1^{(0)})^2 e^{-2r} - \frac{1}{2}(x_2 - x_2^{(0)})^2 e^{2r}\right]$$

The contour of the Wigner function is

$$\frac{(x_1 - x_1^{(0)})^2}{e^{2r}} + \frac{(x_2 - x_2^{(0)})^2}{e^{-2r}} = 1$$

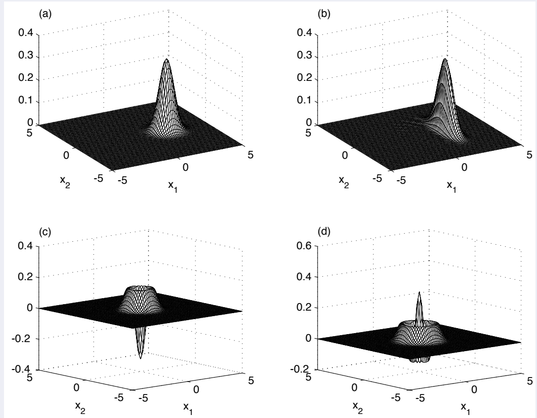
i.e., an ellipse with the lengths of the major and minor axes given by e^r and e^{-r}, respectively.

Number state $|n\rangle$

$$W(\alpha, \alpha^*) = \frac{2}{\pi}(-1)^n \exp(-2|\alpha|^2)\, L_n(4|\alpha|^2)$$

where $L_n(x)$ is the Laguerre polynomial. This Wigner function clearly has negative parts.

Wigner functions



(a) Coherent state $|\alpha = 2\rangle$, (b) squeezed state $|\alpha = 2, r = 0.6\rangle$, (c) number state $|n = 1\rangle$, and (d) the number state $|n = 2\rangle$.

Writing $\hat{a} = (\hat{X}_1 + i\hat{X}_2)/2$ and $\alpha = (x + iy)/2$, one can show that the Wigner function can be rewritten in terms of the matrix elements of $\hat{\rho}$ in the $\hat{X}_1$ representation as

$$W(x, y) = \frac{2}{\pi} \int dx_1' \, \langle x - x_1' | \hat{\rho} | x + x_1' \rangle \, e^{ix_1' y}$$

Hence one can show that

$$\frac{1}{4}\int dy\, W(x, y) = \langle x | \hat{\rho} | x \rangle \quad \text{and} \quad \frac{1}{4}\int dx\, W(x, y) = \langle y | \hat{\rho} | y \rangle$$

i.e., the *probability densities in x and y respectively are obtained by integrating out the other variable*, as for a classical joint probability density.

Optical Homodyne Tomography

RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 40, NUMBER 5 SEPTEMBER 1, 1989

Determination of quasiprobability distributions in terms of probability distributions for the rotated quadrature phase

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(Received 5 June 1989)

It is shown that the probability distribution for the rotated quadrature phase $[a^\dagger \exp(i\theta) + a\exp(-i\theta)]/2$ can be expressed in terms of quasiprobability distributions such as P, Q, and Wigner functions and that also the reverse is true, i.e., if the probability distribution for the rotated quadrature phase is known for every θ in the interval $0 \le \theta < \pi$, then the quasiprobability distributions can be obtained.

Generalised quadrature operators $\hat{X}_\theta = \hat{X}_1 \cos\theta + \hat{X}_2 \sin\theta$, $\hat{P}_\theta = -\hat{X}_1 \sin\theta + \hat{X}_2 \cos\theta$

$$P_\theta(x_\theta) = \frac{1}{4}\int dp_\theta\, W(x_\theta\cos\theta - p_\theta\sin\theta, x_\theta\sin\theta + p_\theta\cos\theta)$$

Given distributions $P_\theta(x_\theta)$ for a finite set of $\theta \in [0, \pi)$, can use *inverse Radon transform* to determine $W(x, y)$.

VOLUME 70, NUMBER 9 PHYSICAL REVIEW LETTERS 1 MARCH 1993

Measurement of the Wigner Distribution and the Density Matrix of a Light Mode Using Optical Homodyne Tomography: Application to Squeezed States and the Vacuum

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(Received 16 November 1992)

We have measured probability distributions of quadrature-field amplitude for both vacuum and quadrature-squeezed states of a mode of the electromagnetic field. From these measurements we demonstrate the technique of optical homodyne tomography to determine the Wigner distribution and the density matrix of the mode. This provides a complete quantum mechanical characterization of the measured mode.

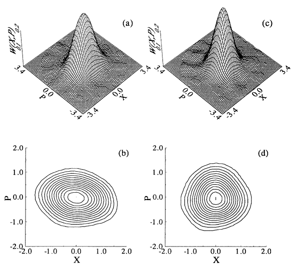
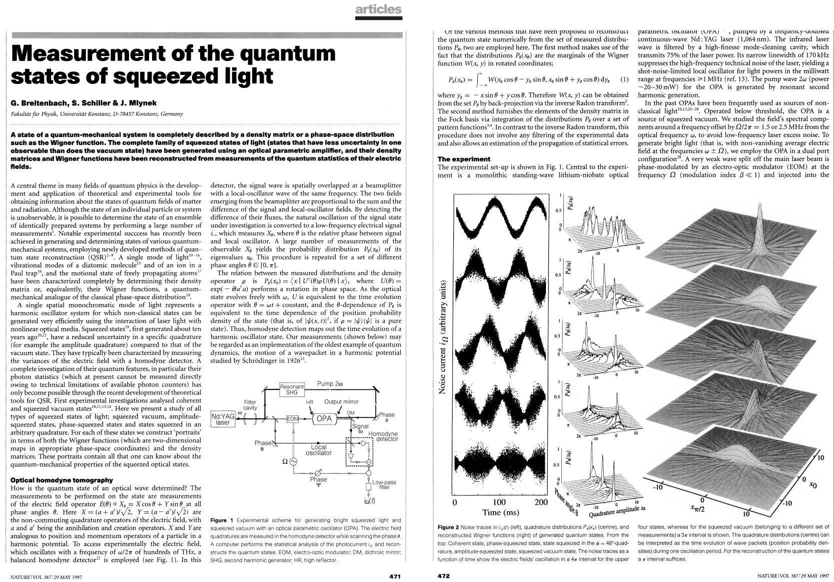


FIG. 1. Measured Wigner distributions for (a),(b) a squeezed state and (c),(d) a vacuum state, viewed in 3D and as contour plots, with equal numbers of constant-height contours. Squeezing of the noise distribution is clearly seen in (b).



articles

Measurement of the quantum states of squeezed light

G. Breitenbach, S. Schiller & J. Mlynek
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A state of a quantum-mechanical system is completely described by a density matrix or a phase-space distribution such as the Wigner function. The complete family of squeezed states of light (states that have less uncertainty in one observable than does the vacuum state) have been generated using an optical parametric amplifier, and their density matrices and Wigner functions have been reconstructed from measurements of the quantum statistics of their electric fields.

VOLUME 87, NUMBER 5 PHYSICAL REVIEW LETTERS 30 JULY 2001

Quantum State Reconstruction of the Single-Photon Fock State

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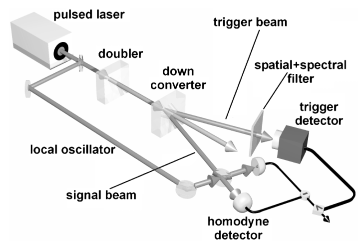
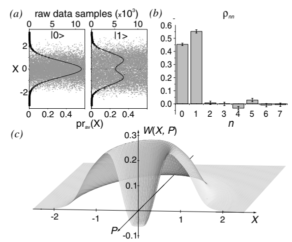
(Received 14 March 2001; published 11 July 2001)

We have reconstructed the quantum state of optical pulses containing single photons using the method of phase-randomized pulsed optical homodyne tomography. The single-photon Fock state $|1\rangle$ was prepared using conditional measurements on photon pairs born in the process of parametric down-conversion. A probability distribution of the phase-averaged electric field amplitudes with a strongly non-Gaussian shape is obtained with the total detection efficiency of $(55 \pm 1)\%$. The angle-averaged Wigner function reconstructed from this distribution shows a strong dip reaching classically impossible negative values around the origin of the phase space.





Theoretical Methods in Quantum Optics 4:
Quantum Phenomena in Simple Nonlinear Optical Systems

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29 September, 2008

Outline

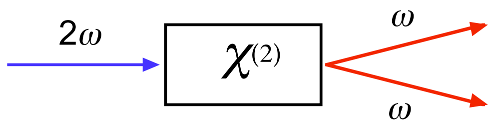
We now consider some simple models of nonlinear optical systems that produce manifestly nonclassical states of light and are classic examples in quantum optics.

Topics

- Degenerate Parametric Amplification
- Non-Degenerate Parametric Amplification

Degenerate Parametric Amplification

- One of the simplest interactions in nonlinear optics is where a photon of frequency 2ω is converted into two photons each with frequency ω.



- This process, known as *parametric down conversion*, may occur in a medium with a second-order nonlinear susceptibility $\chi^{(2)}$ and describes the operation of a *parametric amplifier*.
- In a degenerate parametric amplifier a signal at frequency ω is amplified by pumping a $\chi^{(2)}$ medium (with a laser) at the frequency 2ω.

Model

- Consider a simple model where the *pump mode at frequency 2ω is treated classically* (i.e., the pump field is assumed to be in a large-amplitude coherent state).
- The signal mode at frequency ω is described by the annihilation operator $\hat{a}$.
- The Hamiltonian for the system is then taken to be

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} - \frac{1}{2}i\hbar\chi\left(\hat{a}^2 e^{2i\omega t} - \hat{a}^{\dagger 2}e^{-2i\omega t}\right)$$

where χ is a constant proportional to the second-order nonlinear susceptibility and to the amplitude of the pump field.

In the interaction picture the Hamiltonian becomes

$$\hat{H}_{\mathrm{I}} = -\frac{1}{2} i\hbar\chi \left(\hat{a}^2 - \hat{a}^{\dagger 2} \right)$$

Note: Moving to the interaction picture can be viewed as transforming to a frame rotating at frequency ω.

The Heisenberg equations of motion are

$$\frac{d\hat{a}}{dt} = \frac{1}{i\hbar}[\hat{a}, \hat{H}_{\mathrm{I}}] = \chi \hat{a}^{\dagger}, \qquad \frac{d\hat{a}^{\dagger}}{dt} = \frac{1}{i\hbar}[\hat{a}^{\dagger}, \hat{H}_{\mathrm{I}}] = \chi \hat{a}$$

which have the solution

$$\hat{a}(t) = \hat{a}(0)\cosh(\chi t) + \hat{a}^{\dagger}(0)\sinh(\chi t)$$

which takes the form of the *generator of the squeezing transformation*.

Introducing the quadrature phase operators, $\hat{X}_1 = \hat{a} + \hat{a}^{\dagger}$ and $\hat{X}_2 = -i(\hat{a} - \hat{a}^{\dagger})$ one finds

$$\hat{X}_1(t) = e^{\chi t}\hat{X}_1(0), \qquad \hat{X}_2(t) = e^{-\chi t}\hat{X}_2(0)$$

i.e., the parametric amplifier is a *phase-sensitive amplifier that amplifies one quadrature and attenuates the other*.

The parametric amplifier also reduces (increases) the noise in the $\hat{X}_2$ ($\hat{X}_1$) quadrature. The variances $V(X_i, t)$ satisfy

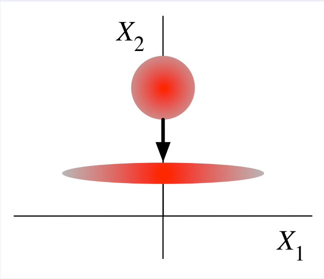
$$V(X_1, t) = e^{2\chi t}V(X_1, 0), \qquad V(X_2, t) = e^{-2\chi t}V(X_2, 0)$$

For initial vacuum or coherent states $V(X_i, 0) = 1$, and hence

$$V(X_1, t) = e^{2\chi t}, \qquad V(X_2, t) = e^{-2\chi t}$$

with the product of the variances satisfying the minimum uncertainty relation, $V(X_1)V(X_2) = 1$.

- Thus, the deamplified quadrature has less quantum noise than the vacuum level.
- The amount of squeezing or noise reduction is proportional to the strength of the nonlinearity, the amplitude of the pump field, and the interaction time.



Non-Degenerate Parametric Amplification

- In the nondegenerate parametric amplifier a pump mode at frequency 2ω interacts in a nonlinear optical medium with two modes at frequencies ω_1 and ω_2, such that $2\omega = \omega_1 + \omega_2$.
- It is conventional to designate one mode as the *signal* and the other as the *idler*.
- Note that in some cases the signal and idler modes may differ in polarisation rather than in frequency.

Model

- Consider again a simple model where the pump mode at frequency 2ω is treated classically.
- The Hamiltonian for this system can be written as

$$\hat{H} = \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + i\hbar\chi \left(\hat{a}_1^\dagger \hat{a}_2^\dagger e^{-2i\omega t} - \hat{a}_1 \hat{a}_2 e^{2i\omega t} \right)$$

  where $\hat{a}_1$ ($\hat{a}_2$) is the annihilation operator for the signal (idler) mode.
- The coupling constant χ is proportional to the second-order susceptibility of the medium and to the (coherent) amplitude of the pump.

The Heisenberg equations of motion in the interaction picture are

$$\frac{d\hat{a}_1}{dt} = \chi \hat{a}_2^\dagger, \qquad \frac{d\hat{a}_2^\dagger}{dt} = \chi \hat{a}_1$$

with solutions

$$\begin{aligned} \hat{a}_1(t) &= \hat{a}_1(0)\cosh(\chi t) + \hat{a}_2^\dagger(0)\sinh(\chi t) \\ \hat{a}_2(t) &= \hat{a}_2(0)\cosh(\chi t) + \hat{a}_1^\dagger(0)\sinh(\chi t) \end{aligned}$$

Note:
These take the form of the *generator of the two-mode squeezing transformation*
– the two-mode squeeze operator is $\hat{S} = \exp[\chi t(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)]$.

Intensity correlations

- The intensity correlation functions of this system exhibit interesting quantum features.
- In particular, with a two-mode system we may consider cross correlations between the two modes and show that *correlations exist that violate classical inequalities*.

Consider the moment $\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle$. We may express this moment in terms of the (two-mode) Glauber-Sudarshan P function as

$$\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle = \int d^2\alpha_1 \int d^2\alpha_2 \, |\alpha_1|^2 |\alpha_2|^2 P(\alpha_1, \alpha_2)$$

If a positive $P(\alpha_1, \alpha_2)$ exists the right-hand-side of this equation is the classical intensity correlation function for two fields with the fluctuating complex amplitudes α_1 and α_2.

The following (Schwarz) inequality then holds:

$$\begin{aligned} &\int d^2\alpha_1 \int d^2\alpha_2 \, |\alpha_1|^2 |\alpha_2|^2 P(\alpha_1, \alpha_2) \\ &\leq \left[\int d^2\alpha_1 \int d^2\alpha_2 \, |\alpha_1|^4 P(\alpha_1, \alpha_2) \right]^{1/2} \\ &\quad \times \left[\int d^2\alpha_1 \int d^2\alpha_2 \, |\alpha_2|^4 P(\alpha_1, \alpha_2) \right]^{1/2} \end{aligned}$$

or, expressed in terms of operators:

$$\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle \leq \left[\langle \hat{a}_1^{\dagger 2} \hat{a}_1^2 \rangle \langle \hat{a}_2^{\dagger 2} \hat{a}_2^2 \rangle \right]^{1/2}$$

This is known as the *Cauchy-Schwarz inequality*. If the two modes are symmetric, then this reduces to

$$\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle \leq \langle \hat{a}_1^{\dagger 2} \hat{a}_1^2 \rangle$$

A stronger inequality may be derived for quantum fields; in particular, from the general result $\mathrm{Tr}(\hat{\rho}\hat{A}^\dagger\hat{A}) \geq 0$ for a linear operator $\hat{A}$ (see earlier), we have

$$\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle^2 \leq \langle (\hat{a}_1^\dagger \hat{a}_1)^2 \rangle \, \langle (\hat{a}_2^\dagger \hat{a}_2)^2 \rangle$$

or, for a symmetrical system,

$$\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle \leq \langle \hat{a}_1^{\dagger 2} \hat{a}_1^2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle$$

So, a violation of the Cauchy-Schwarz inequality is clearly possible in a quantum system.

Consider the nondegenerate parametric amplifier. Because signal and idler photons are always created together, the following conservation law holds:

$$\hat{n}_1(t) - \hat{n}_2(t) = \hat{n}_1(0) - \hat{n}_2(0)$$

Using this relation the intensity correlation function may be written

$$\langle \hat{n}_1(t)\hat{n}_2(t) \rangle = \langle \hat{n}_1(t)^2 \rangle - \langle \hat{n}_1(t)[\hat{n}_1(0) - \hat{n}_2(0)] \rangle$$

For an initial vacuum state the last term is zero, and so

$$\langle \hat{n}_1(t)\hat{n}_2(t) \rangle = \langle \hat{a}_1^\dagger(t)\hat{a}_1^\dagger(t)\hat{a}_1(t)\hat{a}_1(t) \rangle + \langle \hat{a}_1^\dagger(t)\hat{a}_1(t) \rangle$$

which corresponds to the *maximum violation of the Cauchy-Schwarz inequality allowed by quantum mechanics*. Thus, the nondegenerate parametric amplifier exhibits quantum mechanical correlations that violate certain classical inequalities.

A clear experimental demonstration of this violation has been performed, e.g., by Zou *et al.* [Opt. Commun. **84**, 351 (1991)].

Violation of classical probability in parametric down-conversion

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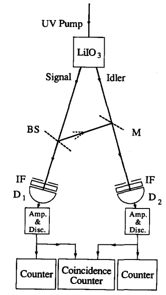
Received 23 April 1991

A classical inequality relating to photoelectric coincidence counting with two light beams is derived. It is then demonstrated in a coincidence counting experiment with the signal and idler photons produced in the process of parametric down-conversion, that the classical inequality is violated by about 600 standard deviations.



Einstein-Podolsky-Rosen (EPR) paradox

- The nondegenerate parametric amplifier can also be used to prepare states of the sort discussed in the EPR paradox.

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

- In the original treatment two systems are prepared in a correlated state.
- One of two canonically conjugate variables is measured on one system and the correlation is such that the value for a physical variable in the second system may be inferred with certainty.

Consider the (generalised) quadrature variables

$$\hat{X}_i^{\theta} = \hat{a}_i e^{i\theta} + \hat{a}_i^{\dagger} e^{-i\theta} \qquad (i = 1, 2)$$

These obey the commutation relation

$$[\hat{X}_i^{\theta}, \hat{X}_i^{\theta+\pi/2}] = -2i$$

and are thus directly analogous to the position and momentum operators discussed in the original EPR paper.

As a measure of the degree of correlation between the two modes, we consider the quantity

$$V(\theta, \phi) = \frac{1}{2}\langle(\hat{X}_1^{\theta} - \hat{X}_2^{\phi})^2\rangle$$

If $V(\theta, \phi) = 0$ then $\hat{X}_1^{\theta}$ is perfectly correlated with $\hat{X}_2^{\phi}$ which means that a measurement of $\hat{X}_1^{\theta}$ can be used to infer a value of $\hat{X}_2^{\phi}$ with certainty.

Using the solutions for the mode operators one finds

$$\begin{aligned} V(\theta, \phi) &= \cosh(2\chi t) - \sinh(2\chi t)\cos(\theta + \phi) \\ &= e^{-2\chi t} \quad \text{for } \theta + \phi = 0 \end{aligned}$$

So, when $\theta + \phi = 0$, for long times $V(\theta, \phi)$ becomes increasingly small, reflecting the build up of correlation between the signal and idler fields. [The initial value $V(\theta, \phi) = 1$ corresponds to uncorrelated systems.]

- So, as time proceeds a measurement of $\hat{X}_1^{\theta}$ yields an increasingly certain value for $\hat{X}_2^{\phi}$.
- However, one could equally well have measured $\hat{X}_1^{\theta-\pi/2}$ which would yield an increasingly certain value for $\hat{X}_2^{\phi+\pi/2}$.
- Thus, certain values for two noncommuting observables, $\hat{X}_2^{\phi}$ and $\hat{X}_2^{\phi+\pi/2}$, may be obtained without in any way disturbing system 2.
- This outcome constitutes the centre of the EPR argument.

- In reality no measurement enables a perfect inference to be made.
- To quantify the extent of the apparent paradox, we can define the variances $V_{\text{inf}}(X_2^{\phi})$ and $V_{\text{inf}}(X_2^{\phi+\pi/2})$ which determine the error in inferring $\hat{X}_2^{\phi}$ and $\hat{X}_2^{\phi+\pi/2}$ from measurements on $\hat{X}_1^{\theta}$ and $\hat{X}_1^{\theta-\pi/2}$.
- In the case of direct measurements made on $(\hat{X}_2^{\phi}, \hat{X}_2^{\phi+\pi/2})$, quantum mechanics would suggest

$$V(X_2^{\phi})V(X_2^{\phi+\pi/2}) \geq 1$$

- However, the variances in the inferred values are not constrained. Thus, whenever

$$V_{\text{inf}}(X_2^{\phi})V_{\text{inf}}(X_2^{\phi+\pi/2}) \leq 1$$

one can claim an EPR correlation paradoxically less than expected by direct measurement on the same state.

Ou *et al.* performed an experimental test of this for a nondegenerate parametric amplifier, obtaining a lowest value of $V_{\rm inf}(X_2^{\phi})\,V_{\rm inf}(X_2^{\phi+\pi/2}) = 0.7 \pm 0.01$.

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Realization of the Einstein-Podolsky-Rosen Paradox for Continuous Variables

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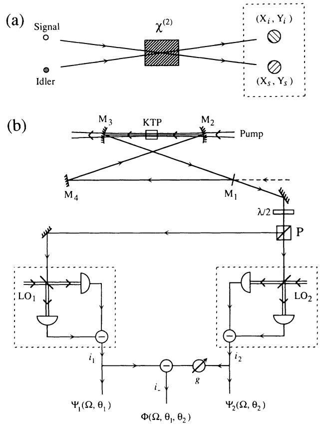
(Received 20 February 1992)

The Einstein-Podolsky-Rosen paradox is demonstrated experimentally for dynamical variables having a continuous spectrum. As opposed to previous work with discrete spin or polarization variables, the continuous optical amplitudes of a signal beam are inferred in turn from those of a spatially separated but strongly correlated idler beam generated by nondegenerate parametric amplification. The uncertainty product for the variances of these inferences is observed to be 0.70 ± 0.01, which is below the limit of unity required for the demonstration of the paradox.



Wigner function

- The full quantum correlations present in the parametric amplifier may be represented using a quasiprobability distribution.
- If both modes of the amplifier are initially in the vacuum state no Glauber-Sudarshan *P* function for the total system exists at any time.
- However, a Wigner function does exist.

The appropriate two-mode characteristic function is given by

$$\begin{aligned}\chi_S(z_1, z_2, t) &= \langle 0,0|e^{iz_1^*\hat{a}_1^\dagger(t)+iz_1\hat{a}_1(t)}e^{iz_2^*\hat{a}_2^\dagger(t)+iz_2\hat{a}_2(t)}|0,0\rangle \\ &= e^{-\frac{1}{2}|z_1(t)|^2-\frac{1}{2}|z_2(t)|^2}\end{aligned}$$

where

$$\begin{aligned}z_1(t) &= z_1^*\cosh(\chi t) + z_2\sinh(\chi t) \\ z_2(t) &= z_2^*\cosh(\chi t) + z_1\sinh(\chi t)\end{aligned}$$

The Wigner function is then

$$\begin{aligned}W(\alpha_1, \alpha_2, t) &= \frac{1}{\pi^4}\int d^2z_1 \int d^2z_2\, e^{-iz_1^*\alpha_1^*-iz_1\alpha_1}e^{-iz_2^*\alpha_2^*-iz_2\alpha_2}\,\chi_S(z_1, z_2, t) \\ &= \frac{4}{\pi^2}\exp\Big[-2|\alpha_1\cosh(\chi t) - \alpha_2^*\sinh(\chi t)|^2 \\ &\qquad -2|\alpha_2\cosh(\chi t) - \alpha_1^*\sinh(\chi t)|^2\Big] \\ &= \frac{4}{\pi^2}\exp\left[-\frac{1}{2}\left(\frac{|\alpha_1+\alpha_2^*|^2}{e^{2\chi t}} + \frac{|\alpha_1-\alpha_2^*|^2}{e^{-2\chi t}}\right)\right]\end{aligned}$$

which shows that squeezing occurs in a linear combination of the two modes. Note also the following limit, with $\alpha_j = x_j + iy_j$,

$$W(x_1, y_1, x_2, y_2) \to C\,\delta(x_1 - x_2)\,\delta(y_1 + y_2) \quad \text{as} \quad \chi t \to \infty$$

which corresponds precisely to the state originally envisioned by EPR.

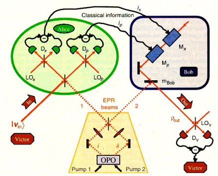
RESEARCH ARTICLES

Unconditional Quantum Teleportation

A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble,* E. S. Polzik

Quantum teleportation of optical coherent states was demonstrated experimentally using squeezed-state entanglement. The quantum nature of the achieved teleportation was verified by the experimentally determined fidelity $F^{\rm exp} = 0.58 \pm 0.02$, which describes the match between input and output states. A fidelity greater than 0.5 is not possible for coherent states without the use of entanglement. This is the first realization of unconditional quantum teleportation where every state entering the device is actually teleported.

Quantum teleportation is the disembodied transport of an unknown quantum state from one place to another (1). All protocols for accomplishing such transport require nonlocal correlations, or entanglement, between systems shared by the sender and receiver. John Bell's famous theorem on the incompatibility of quantum mechanics with local hid-



706 23 OCTOBER 1998 VOL 282 SCIENCE www.sciencemag.org

Theoretical Methods in Quantum Optics 5:
Master Equation Methods I

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29 September, 2008

Outline

In all physical processes there is an associated loss mechanism. In the context of quantum optics, specific sources of loss include, e.g., imperfect mirrors and atomic spontaneous emission. We now consider one particular way of including losses in the quantum mechanical equations of motion – the *master equation approach*. In this approach, the system of interest is coupled to a *heat bath* or *reservoir*, which describes the environment into which the system loses energy.

Topics

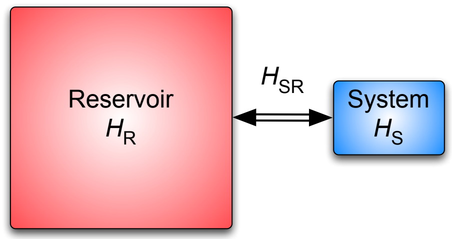
- The Master Equation
- System Operator Expectation Values
- Correlation Functions: Quantum Regression Formula

The Master Equation

We begin with a Hamiltonian of the general form

$$\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR}$$

- $\hat{H}_S$, $\hat{H}_R$ are Hamiltonians for the system and reservoir.
- $\hat{H}_{SR}$ describes the interaction between them.



Let $\hat{w}(t)$ be the density operator for the total system $S \oplus R$.

We define the *reduced density operator* $\hat{\rho}(t) = Tr_R[\hat{w}(t)]$, where the trace is only taken over the reservoir states.

If $\hat{O}$ is an operator in S we can calculate its average in the Schrödinger picture if we have knowledge of $\hat{\rho}(t)$ alone, i.e.,

$$\langle\hat{O}\rangle = \text{Tr}_{S\oplus R}[\hat{O}\hat{w}(t)] = \text{Tr}_S\{\hat{O}\text{Tr}_R[\hat{w}(t)]\} = \text{Tr}_S[\hat{O}\hat{\rho}(t)]$$

Our objective is to obtain an equation for $\hat{\rho}(t)$ with the properties of the reservoir R entering only as parameters.

- The Schrödinger equation for $\hat{w}(t)$ is

$$\dot{\hat{w}}(t) = \frac{1}{i\hbar}[\hat{H}, \hat{w}(t)]$$

- Transform into the interaction picture,

$$\tilde{w}(t) = e^{i(\hat{H}_S+\hat{H}_R)t/\hbar}\hat{w}(t)e^{-i(\hat{H}_S+\hat{H}_R)t/\hbar}$$

to give

$$\dot{\tilde{w}}(t) = \frac{1}{i\hbar}[\tilde{H}_{SR}(t), \tilde{w}(t)]$$

where now $\tilde{H}_{SR}(t)$ is explicitly time-dependent:

$$\tilde{H}_{SR}(t) = e^{i(\hat{H}_S+\hat{H}_R)t/\hbar}\hat{H}_{SR}e^{-i(\hat{H}_S+\hat{H}_R)t/\hbar}$$

- Now integrate formally to give

$$\tilde{w}(t) = \tilde{w}(0) + \frac{1}{i\hbar}\int_0^t dt' \left[\tilde{H}_{SR}(t'), \tilde{w}(t')\right]$$

- Substitute this expression for $\tilde{w}(t)$ into original equation:

$$\dot{\tilde{w}}(t) = \frac{1}{i\hbar}[\tilde{H}_{SR}(t), \tilde{w}(0)] - \frac{1}{\hbar^2}\int_0^t dt' \left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(t'), \tilde{w}(t')\right]\right]$$

- This equation is exact, and in this form we can identify reasonable approximations to make.

Assumption

We assume that the interaction is turned on at $t = 0$ and that no correlations exist between S and R at this initial time. Then

$$\hat{w}(0) = \tilde{w}(0) = \hat{\rho}(0)\hat{R}_0$$

where $\hat{R}_0$ is an initial reservoir density operator.

- Then, noting that

$$\begin{aligned} \mathrm{Tr}_R[\tilde{w}(t)] &= e^{i\hat{H}_S t/\hbar}\mathrm{Tr}_R[e^{i\hat{H}_R t/\hbar}\hat{w}(t)e^{-i\hat{H}_R t/\hbar}]e^{-i\hat{H}_S t/\hbar} \\ &= e^{i\hat{H}_S t/\hbar}\hat{\rho}(t)\,e^{-i\hat{H}_S t/\hbar} = \tilde{\rho}(t) \end{aligned}$$

tracing over the reservoir gives

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_R\left\{\left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(t'), \tilde{w}(t')\right]\right]\right\}$$

Note

For simplicity, we have eliminated the term $(1/i\hbar)\mathrm{Tr}_R\{[\tilde{H}_{SR}(t), \hat{w}(0)]\}$ with the assumption that

$$\mathrm{Tr}_R[\tilde{H}_{SR}(t)\hat{R}_0] = 0$$

This is guaranteed if the reservoir operators coupling to S have zero mean in the state $\hat{R}_0$ – this can always be arranged by simply including $\mathrm{Tr}_R(\tilde{H}_{SR}\hat{R}_0)$ in the system Hamiltonian $\hat{H}_S$.

- While we have assumed that $\tilde{w}$ factorises at $t = 0$, at later times correlations between S and R may arise due to their coupling through $\hat{H}_{SR}$.
- However, we also assume that this coupling is very weak, and at all times $\hat{w}(t)$ should only show deviations of order $\hat{H}_{SR}$ from an uncorrelated state.
- Furthermore, *R is a large system whose state should be virtually unaffected by its coupling to S*. We therefore write

$$\tilde{w}(t) = \tilde{\rho}(t)\hat{R}_0 + O(\hat{H}_{SR})$$

Born approximation

Neglecting terms higher than second order in $\hat{H}_{SR}$, we write

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2}\int_0^t dt' \, \mathrm{Tr}_R \left\{ \left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(t'), \tilde{\rho}(t')\hat{R}_0\right]\right] \right\}$$

This is still a complicated equation. In particular, it is not Markovian since the future evolution of $\tilde{\rho}(t)$ depends on its past history through the integration over $\tilde{\rho}(t')$ (the future behaviour of a Markovian system depends only on its present state).

Markov approximation

We replace $\tilde{\rho}(t')$ by $\tilde{\rho}(t)$ to obtain a *master equation in the Born-Markov approximation*:

$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2}\int_0^t dt' \, \mathrm{Tr}_R \left\{ [\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\rho}(t)\hat{R}_0]] \right\}$$

Note

Markovian behaviour seems reasonable on physical grounds.

- Potentially, S can depend on its past history because its earlier states become imprinted as changes in the reservoir state (through $\hat{H}_{SR}$) and are then reflected back on the future evolution of S as it interacts with the changed reservoir.
- If, however, the reservoir is a large system maintained in thermal equilibrium, we do not expect it to preserve the minor changes brought about by its interaction with S for very long; not for long enough to significantly affect the future evolution of S.
- It is a question of *reservoir correlation time versus the time scale for significant change in S*.

Let us consider a more specific model:

$$\hat{H}_{SR} = \hbar \sum_i \hat{s}_i \hat{\Gamma}_i \quad \text{or} \quad \tilde{H}_{SR}(t) = \hbar \sum_i \tilde{s}_i(t)\tilde{\Gamma}_i(t)$$

where $\{\hat{s}_i\}$ are operators in the Hilbert space of S and $\{\hat{\Gamma}_i\}$ are operators in the Hilbert space of R. In the Born approximation

$$\begin{aligned}
\dot{\tilde{\rho}}(t) &= -\sum_{i,j}\int_0^t dt' \, \mathrm{Tr}_R \left\{ \left[\tilde{s}_i(t)\tilde{\Gamma}_i(t), \left[\tilde{s}_j(t')\tilde{\Gamma}_j(t'), \tilde{\rho}(t')\hat{R}_0\right]\right] \right\} \\
&= -\sum_{i,j}\int_0^t dt' \left[\tilde{s}_i(t)\tilde{s}_j(t')\tilde{\rho}(t') - \tilde{s}_j(t')\tilde{\rho}(t')\tilde{s}_i(t)\right] \langle \tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_R \\
&\quad -\sum_{i,j}\int_0^t dt' \left[\tilde{\rho}(t')\tilde{s}_j(t')\tilde{s}_i(t) - \tilde{s}_i(t)\tilde{\rho}(t')\tilde{s}_j(t')\right] \langle \tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\rangle_R
\end{aligned}$$

where we have used the cyclic property of the trace, i.e., $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \mathrm{Tr}(\hat{C}\hat{A}\hat{B}) = \mathrm{Tr}(\hat{B}\hat{C}\hat{A})$.

The properties of the reservoir enter through the correlation functions

$$\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_{\rm R} = {\rm tr}_{\rm R}\left[\hat{R}_0\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\right], \quad \langle\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\rangle_{\rm R} = {\rm tr}_{\rm R}\left[\hat{R}_0\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\right]$$

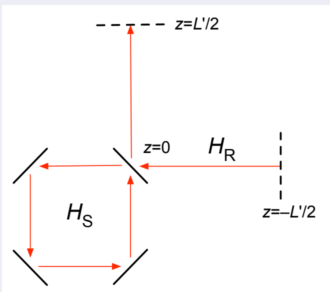
- We can justify the replacement of $\tilde{\rho}(t')$ by $\tilde{\rho}(t)$ if these correlation functions decay very rapidly on the time scale on which $\tilde{\rho}(t)$ varies; e.g., if

$$\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\rangle_{\rm R} \sim \delta(t-t')$$

- So, the Markov approximation relies on the existence of *two widely separated time scales*: a slow time scale for the dynamics of the system S, and a fast time scale characterising the decay of reservoir correlation functions.

Master equation for a cavity mode driven by thermal light

- Consider a ring cavity with the reservoir comprised of travelling-wave modes that satisfy periodic boundary conditions at $z=-L'/2$ and $z=L'/2$.
- The (single) cavity mode, system S, couples to the reservoir through a partially transmitting mirror at $z=0$.



Hamiltonians:

$$\begin{aligned}\hat{H}_{\rm S} &= \hbar\omega_{\rm c}\hat{a}^\dagger\hat{a}\\ \hat{H}_{\rm R} &= \sum_j \hbar\omega_j\hat{r}_j^\dagger\hat{r}_j\\ \hat{H}_{\rm SR} &= \sum_j \hbar\left(\kappa_j^*\hat{a}\hat{r}_j^\dagger + \kappa_j\hat{a}^\dagger\hat{r}_j\right) = \hbar\left(\hat{a}\hat{\Gamma}^\dagger + \hat{a}^\dagger\hat{\Gamma}\right)\end{aligned}$$

- The system S is a harmonic oscillator with frequency $\omega_{\rm c}$ and annihilation operator $\hat{a}$.
- The reservoir is a collection of harmonic oscillators with frequencies ω_j and annihilation operators $\hat{r}_j$. These reservoir oscillators couple to the cavity mode oscillator with coupling constants κ_j.
- The interaction is modelled in the *rotating-wave approximation*. This amounts to neglecting terms of the form $\hat{a}\hat{\Gamma}$ or $\hat{a}^\dagger\hat{\Gamma}^\dagger$, which are *energy non-conserving*.

The reservoir is taken to be in thermal equilibrium at temperature T, so

$$\hat{R}_0 = \prod_j e^{-\hbar\omega_j\hat{r}_j^\dagger\hat{r}_j/k_{\rm B}T}\left(1 - e^{-\hbar\omega_j/k_{\rm B}T}\right)$$

where $k_{\rm B}$ is Boltzmann's constant.

The interaction Hamiltonian corresponds to

$$\hat{s}_1 = \hat{a}, \qquad \hat{s}_2 = \hat{a}^\dagger$$
$$\hat{\Gamma}_1 = \hat{\Gamma}^\dagger = \sum_j \kappa_j^*\hat{r}_j^\dagger, \qquad \hat{\Gamma}_2 = \hat{\Gamma} = \sum_j \kappa_j\hat{r}_j$$

and in the interaction picture

$$\tilde{s}_1(t) = e^{i\omega_c\hat{a}^\dagger\hat{a}t}\hat{a}\,e^{-i\omega_c\hat{a}^\dagger\hat{a}t} = \hat{a}\,e^{-i\omega_c t}, \qquad \tilde{s}_2(t) = \hat{a}^\dagger e^{i\omega_c t}$$
$$\tilde{\Gamma}_1(t) = \tilde{\Gamma}^\dagger(t) = \sum_j \kappa_j^*\hat{r}_j^\dagger e^{i\omega_j t}, \qquad \tilde{\Gamma}_2(t) = \tilde{\Gamma}(t) = \sum_j \kappa_j\hat{r}_j e^{-i\omega_j t}$$

The master equation in the Born approximation is then

$$\begin{aligned}\dot{\tilde\rho}(t) &= -\int_0^t dt' \Big\{ [\hat a\hat a\tilde\rho(t') - \hat a\tilde\rho(t')\hat a]\, e^{-i\omega_c(t+t')}\langle\tilde\Gamma^\dagger(t)\tilde\Gamma^\dagger(t')\rangle_R + \text{h.c.}\\ &\quad + [\hat a^\dagger\hat a^\dagger\tilde\rho(t') - \hat a^\dagger\tilde\rho(t')\hat a^\dagger]\, e^{i\omega_c(t+t')}\langle\tilde\Gamma(t)\tilde\Gamma(t')\rangle_R + \text{h.c.}\\ &\quad + [\hat a\hat a^\dagger\tilde\rho(t') - \hat a^\dagger\tilde\rho(t')\hat a]\, e^{-i\omega_c(t-t')}\langle\tilde\Gamma^\dagger(t)\tilde\Gamma(t')\rangle_R + \text{h.c.}\\ &\quad + [\hat a^\dagger\hat a\tilde\rho(t') - \hat a\tilde\rho(t')\hat a^\dagger]\, e^{i\omega_c(t-t')}\langle\tilde\Gamma(t)\tilde\Gamma^\dagger(t')\rangle_R + \text{h.c.}\Big\}\end{aligned}$$

where the reservoir correlation functions are explicitly:

$$\begin{aligned}\langle\tilde\Gamma^\dagger(t)\tilde\Gamma^\dagger(t')\rangle_R &= \langle\tilde\Gamma(t)\tilde\Gamma(t')\rangle_R = 0\\ \langle\tilde\Gamma^\dagger(t)\tilde\Gamma(t')\rangle_R &= \sum_j |\kappa_j|^2 e^{i\omega_j(t-t')}\bar n(\omega_j,T)\\ \langle\tilde\Gamma(t)\tilde\Gamma^\dagger(t')\rangle_R &= \sum_j |\kappa_j|^2 e^{-i\omega_j(t-t')}[\bar n(\omega_j,T)+1]\\ \text{with}\quad \bar n(\omega_j,T) &= \text{Tr}_R\left(\hat R_0\hat r_j^\dagger\hat r_j\right) = \frac{e^{-\hbar\omega_j/k_BT}}{1-e^{-\hbar\omega_j/k_BT}} = \frac{1}{e^{\hbar\omega_j/k_BT}-1}\end{aligned}$$

Integral representation

Introduce a density of states $g(\omega)$, such that $g(\omega)d\omega$ = number of oscillators with frequencies in the interval $(\omega, \omega+d\omega)$. For the 1-d reservoir field we are considering,

$$g(\omega) = L'/(2\pi c)$$

Defining $\tau = t - t'$, we can then write the reservoir correlation functions in integral form as

$$\begin{aligned}\langle\tilde\Gamma^\dagger(t)\tilde\Gamma(t-\tau)\rangle_R &= \int_0^\infty d\omega\, e^{i\omega\tau}g(\omega)|\kappa(\omega)|^2\bar n(\omega,T)\\ \langle\tilde\Gamma(t)\tilde\Gamma^\dagger(t-\tau)\rangle_R &= \int_0^\infty d\omega\, e^{-i\omega\tau}g(\omega)|\kappa(\omega)|^2[\bar n(\omega,T)+1]\end{aligned}$$

Markov approximation

- To estimate the reservoir correlation time, take $\kappa(\omega) \simeq$ constant and consider the frequency dependence of $\bar n(\omega, T)$.
- Because of the factor $e^{\pm i\omega_c\tau}$ multiplying the reservoir correlation functions in $\dot{\tilde\rho}(t)$, it is really only the $\omega \approx \omega_c$ part of the frequency range that is important.
- Can therefore estimate the reservoir correlation time by extending the frequency integrals to $-\infty$ [with $\bar n(\omega,T) \to \bar n(|\omega|,T)$].
- One then has a Fourier transform and the *correlation time is given by the inverse width $\hbar/k_BT$* of the function $\bar n(|\omega|, T)$.
- At room temperature this gives a number of the order of 0.25×10^{-13}*sec* $\ll$ *time scale for significant changes in* $\tilde\rho$ (a typical decay time for an optical cavity mode $\sim 10^{-8}$sec).

So, we can replace $\tilde\rho(t-\tau)$ by $\tilde\rho(t)$ in the integrals. Then

$$\dot{\tilde\rho} = \alpha\left(\hat a\tilde\rho\hat a^\dagger - \hat a^\dagger\hat a\tilde\rho\right) + \beta\left(\hat a\tilde\rho\hat a^\dagger + \hat a^\dagger\tilde\rho\hat a - \hat a^\dagger\hat a\tilde\rho - \tilde\rho\hat a\hat a^\dagger\right) + \text{h.c.}$$

where $\tilde\rho \equiv \tilde\rho(t)$, with

$$\begin{aligned}\alpha &= \int_0^t d\tau \int_0^\infty d\omega\, e^{-i(\omega-\omega_c)\tau}g(\omega)|\kappa(\omega)|^2\\ \beta &= \int_0^t d\tau \int_0^\infty d\omega\, e^{-i(\omega-\omega_c)\tau}g(\omega)|\kappa(\omega)|^2\bar n(\omega,T)\end{aligned}$$

- Now, t is a time typical of the time scale for changes in $\tilde\rho$, while the τ integration is dominated by much shorter times characterising the decay of reservoir correlations.

- So, we can extend the τ integration to infinity and use

$$\lim_{t\to\infty}\int_0^t d\tau\, e^{-i(\omega-\omega_c)\tau} = \pi\delta(\omega-\omega_c) + i\,\frac{P}{\omega_c-\omega}$$

where P indicates the Cauchy principal value. This gives

$$\begin{aligned}\alpha &= \pi g(\omega_c)|\kappa(\omega_c)|^2 + i\Delta\\ \beta &= \pi g(\omega_c)|\kappa(\omega_c)|^2\bar{n}(\omega_c) + i\Delta'\end{aligned}$$

with

$$\Delta = P\int_0^\infty d\omega\,\frac{g(\omega)|\kappa(\omega)|^2}{\omega_c-\omega},\qquad \Delta' = P\int_0^\infty d\omega\,\frac{g(\omega)|\kappa(\omega)|^2}{\omega_c-\omega}\,\bar{n}(\omega,T)$$

- Define

$$\kappa = \pi g(\omega_c)|\kappa(\omega_c)|^2,\qquad \bar{n} = \bar{n}(\omega_c,T)$$

We finally obtain our master equation:

$$\begin{aligned}\dot{\tilde{\rho}} &= -i\Delta[\hat{a}^\dagger\hat{a},\tilde{\rho}] + \kappa\left(2\hat{a}\tilde{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\tilde{\rho} - \tilde{\rho}\hat{a}^\dagger\hat{a}\right)\\ &\quad + 2\kappa\bar{n}\left(\hat{a}\tilde{\rho}\hat{a}^\dagger + \hat{a}^\dagger\tilde{\rho}\hat{a} - \hat{a}^\dagger\hat{a}\tilde{\rho} - \tilde{\rho}\hat{a}\hat{a}^\dagger\right)\end{aligned}$$

Transform back to the Schrödinger picture using

$$\dot{\hat{\rho}} = \frac{1}{i\hbar}[\hat{H}_S,\hat{\rho}] + e^{-i\hat{H}_S t/\hbar}\dot{\tilde{\rho}}\,e^{i\hat{H}_S t/\hbar}$$

Master equation for a cavity mode driven by thermal light

$$\begin{aligned}\dot{\hat{\rho}} = -i\omega_c'[\hat{a}^\dagger\hat{a},\hat{\rho}] \;&+\; \kappa(\bar{n}+1)\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)\\ &+\; \kappa\bar{n}\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right)\end{aligned}$$

where $\omega_c' = \omega_c + \Delta$.

System Operator Expectation Values

- Equations of motion for the expectation values of system operators may be derived directly from the master equation.
- For example, the evolution of the mean amplitude of the cavity mode, $\langle\hat{a}\rangle$, is given by

$$\begin{aligned}\langle\dot{\hat{a}}\rangle &= \mathrm{Tr}(\hat{a}\dot{\hat{\rho}})\\ &= -i\omega_c'\,\mathrm{Tr}(\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}) + \kappa(\bar{n}+1)\,\mathrm{Tr}(2\hat{a}^2\hat{\rho}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{a}\hat{\rho} - \hat{a}\hat{\rho}\hat{a}^\dagger\hat{a})\\ &\quad + \kappa\bar{n}\,\mathrm{Tr}(2\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}^2\hat{a}^\dagger\hat{\rho} - \hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger)\\ &= -i\omega_c'\,\mathrm{Tr}[(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})\hat{a}\hat{\rho}] + \kappa(\bar{n}+1)\,\mathrm{Tr}[(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger)\hat{a}\hat{\rho}]\\ &\quad + \kappa\bar{n}\,\mathrm{Tr}[\hat{a}(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})\hat{\rho}]\\ &= -(\kappa + i\omega_c')\langle\hat{a}\rangle\end{aligned}$$

Hence, the mean amplitude decays at a rate κ.

- The mean number of quanta, $\langle\hat{n}\rangle = \langle\hat{a}^\dagger\hat{a}\rangle$, obeys the equation

$$\langle\dot{\hat{n}}\rangle = -2\kappa(\langle\hat{n}\rangle - \bar{n})$$

with solution

$$\langle\hat{n}(t)\rangle = \langle\hat{n}(0)\rangle e^{-2\kappa t} + \bar{n}(1 - e^{-2\kappa t})$$

Thermal fluctuations are “fed” into the cavity from the reservoir; the mean energy does not decay to zero but to the mean energy for a harmonic oscillator with frequency ω_c in thermal equilibrium at temperature T.

Correlation Functions: Quantum Regression Formula

Remaining with the example of a single (cavity) field mode, correlation functions of particular interest are

$$\begin{aligned} G^{(1)}(t,t+\tau) &\propto \langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle \\ G^{(2)}(t,t+\tau) &\propto \langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle \end{aligned}$$

- The *first-order correlation function* is required for calculating the *spectrum* of the field.
- The *second-order correlation function* gives information about the *photon statistics* (e.g., describes photon bunching or antibunching).
- Note that while we would normally associate a single mode with a single frequency, here we are considering a mode defined in a lossy optical cavity, which therefore has a finite linewidth.

Note:
The master equation for the reduced density operator $\hat{\rho}$ can be written formally as

$$\dot{\hat{\rho}} = \mathcal{L}\hat{\rho}$$

with formal solution $\hat{\rho}(t) = e^{\mathcal{L}t}\hat{\rho}(0)$.

Here $\mathcal{L}$ is a generalised Liouvillian, or "superoperator"; $\mathcal{L}$ operates on operators rather than on states.

For the damped harmonic oscillator, the action of $\mathcal{L}$ on an arbitrary operator $\hat{O}$ is defined by

$$\begin{aligned} \mathcal{L}\hat{O} \equiv & -i\omega_0[\hat{a}^\dagger\hat{a},\hat{O}] + \kappa\left(2\hat{a}\hat{O}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{O} - \hat{O}\hat{a}^\dagger\hat{a}\right) \\ & + 2\kappa\bar{n}\left(\hat{a}\hat{O}\hat{a}^\dagger + \hat{a}^\dagger\hat{O}\hat{a} - \hat{a}^\dagger\hat{a}\hat{O} - \hat{O}\hat{a}\hat{a}^\dagger\right) \end{aligned}$$

Quantum regression formula

In the Born-Markov approximation, one can derive the following formal expressions for the two-time correlation functions ($\tau \geq 0$):

$$\langle \hat{O}_1(t)\hat{O}_2(t+\tau)\rangle = \mathrm{Tr}_S\left\{\hat{O}_2(0)e^{\mathcal{L}\tau}\left[\hat{\rho}(t)\hat{O}_1(0)\right]\right\}$$

$$\langle \hat{O}_1(t+\tau)\hat{O}_2(t)\rangle = \mathrm{Tr}_S\left\{\hat{O}_1(0)e^{\mathcal{L}\tau}\left[\hat{\rho}(t)\hat{O}_2(0)\right]\right\}$$

$$\langle \hat{O}_1(t)\hat{O}_2(t+\tau)\hat{O}_3(t)\rangle = \mathrm{Tr}_S\left\{\hat{O}_2(0)e^{\mathcal{L}\tau}\left[\hat{O}_3(0)\hat{\rho}(t)\hat{O}_1(0)\right]\right\}$$

Note:
The 1st and 2nd equations are just special cases of the 3rd formula, with either $\hat{O}_1$ or $\hat{O}_3$ set equal to the unit operator.

Quantum regression formula for a complete set of operators

A more convenient form of the quantum regression theorem exists which directly relates the equations of motion for two-time correlation functions to the equations of motion for one-time averages of system operators.

We assume that there exists a complete set of system operators $\hat{A}_\mu$, $\mu = 1, 2, \ldots$, in the sense that we can write

$$\langle \dot{\hat{A}}_\mu \rangle = \mathrm{Tr}_S(\hat{A}_\mu\dot{\hat{\rho}}) = \sum_\lambda M_{\mu\lambda}\langle \hat{A}_\lambda\rangle$$

where the $M_{\mu\lambda}$ are constants. Thus, the expectation values $\langle \hat{A}_\mu\rangle$ obey a coupled set of linear equations with the evolution matrix **M** defined by the elements $M_{\mu\lambda}$. In vector notation,

$$\langle \dot{\hat{\mathbf{A}}}\rangle = \mathbf{M}\langle \mathbf{A}\rangle$$

where $\hat{\mathbf{A}}$ is the column vector of operators $\{\hat{A}_\mu\}$.

Using the formal expression of the quantum regression formula,

$$\begin{aligned}\frac{d}{d\tau}\langle \hat{O}_1(t)\hat{A}_\mu(t+\tau)\rangle &= \mathrm{Tr}_S\left\{\hat{A}_\mu(0)\left(\mathcal{L}e^{\mathcal{L}\tau}[\hat{\rho}(t)\hat{O}_1(0)]\right)\right\}\\ &= \sum_\lambda M_{\mu\lambda}\mathrm{Tr}_S\left\{\hat{A}_\lambda(0)\left(e^{\mathcal{L}\tau}[\hat{\rho}(t)\hat{O}_1(0)]\right)\right\}\\ &= \sum_\lambda M_{\mu\lambda}\langle \hat{O}_1(t)\hat{A}_\lambda(t+\tau)\rangle\end{aligned}$$

or $$\frac{d}{d\tau}\langle \hat{O}_1(t)\hat{\mathbf{A}}(t+\tau)\rangle = \mathbf{M}\langle \hat{O}_1(t)\hat{\mathbf{A}}(t+\tau)\rangle$$

where $\hat{O}_1$ can be any system operator, not necessarily one of the $\hat{A}_\mu$.

Hence, for each operator $\hat{O}_1$, *the set of correlation functions* $\{\langle \hat{O}_1(t)\hat{A}_\mu(t+\tau)\rangle\}$, with $\tau \geq 0$, *satisfies the same equations (as functions of τ) as do the averages $\langle \hat{A}_\mu(t+\tau)\rangle$.*

Similarly, one can show ($\tau \geq 0$)

$$\frac{d}{d\tau}\langle \hat{\mathbf{A}}(t+\tau)\hat{O}_2(t)\rangle = \mathbf{M}\langle \hat{\mathbf{A}}(t+\tau)\hat{O}_2(t)\rangle$$

and

$$\frac{d}{d\tau}\langle \hat{O}_1(t)\hat{\mathbf{A}}(t+\tau)\hat{O}_2(t)\rangle = \mathbf{M}\langle \hat{O}_1(t)\hat{\mathbf{A}}(t+\tau)\hat{O}_2(t)\rangle$$

Correlation functions for the damped harmonic oscillator

For the mean oscillator amplitude we have

$$\langle \dot{\hat{a}}\rangle = -(i\omega_0+\kappa)\langle \hat{a}\rangle$$

Then, with $\hat{A}_1 = \hat{a}$ and $\hat{O}_1 = \hat{a}^\dagger$, we may write

$$\frac{d}{d\tau}\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = -(i\omega_0+\kappa)\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle$$

and thus

$$\begin{aligned}\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle &= \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle\, e^{-(i\omega_0+\kappa)\tau}\\ &= \left[\langle \hat{n}(0)\rangle e^{-\kappa t} + \bar{n}(1-e^{-2\kappa t})\right] e^{-(i\omega_0+\kappa)\tau}\end{aligned}$$

In the long-time (stationary) limit

$$\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle_{ss} \equiv \lim_{t\to\infty}\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle = \bar{n}\, e^{-(i\omega_0+\kappa)\tau}$$

The Fourier transform of this correlation function gives the spectrum of the light at the cavity output, which is simply a Lorentzian with full-width at half-maximum 2κ.

Similarly, in the stationary limit

$$\begin{aligned}\langle \hat{a}^\dagger(0)\hat{a}^\dagger(\tau)\hat{a}(\tau)\hat{a}(0)\rangle &\equiv \lim_{t\to\infty}\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle\\ &= \bar{n}^2(1+e^{-2\kappa\tau})\end{aligned}$$

This expression describes the *photon bunching* associated with thermal light; at zero delay ($\tau = 0$) the correlation function has twice the value it has for long delays ($\kappa\tau \gg 1$).

Theoretical Methods in Quantum Optics 5:
Master Equation Methods II

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Outline

Using the quasiprobability representations for the density operator introduced earlier, the operator master equation can often be converted into a *c*-number Fokker-Planck equation, for which stationary and time-dependent solutions may sometimes be found.

Topics

- Equivalent *c*-Number Equations
- Stochastic Differential Equations
- Limitations

Equivalent *c*-Number Equations

Glauber-Sudarshan representation

An operator master equation may be transformed to a *c*-number equation using the Glauber-Sudarshan representation for $\hat{\rho}$.

Consider again the damped harmonic oscillator:

$$\begin{aligned}\dot{\hat{\rho}} &= -i\omega_0[\hat{a}^\dagger\hat{a},\hat{\rho}] + \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) \\ &\quad + 2\kappa\bar{n}\left(\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right)\end{aligned}$$

We substitute the diagonal representation for $\hat{\rho}$,

$$\hat{\rho} = \int d^2\alpha\, |\alpha\rangle\langle\alpha|\, P(\alpha)$$

The action of the operators $\hat{a}$ and $\hat{a}^\dagger$ on $|\alpha\rangle\langle\alpha|$ (from both the right and left) is replaced by multiplication by the complex variables α and α^*, and by the action of partial derivatives with respect to these variables.

This is achieved using $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and the results

$$\begin{aligned}\frac{\partial}{\partial\alpha}|\alpha\rangle\langle\alpha| &= \frac{\partial}{\partial\alpha}\left(e^{-|\alpha|^2}e^{\alpha\hat{a}^\dagger}|0\rangle\langle 0|e^{\alpha^*\hat{a}}\right) = \left(\hat{a}^\dagger - \alpha^*\right)|\alpha\rangle\langle\alpha| \\ \frac{\partial}{\partial\alpha^*}|\alpha\rangle\langle\alpha| &= \frac{\partial}{\partial\alpha^*}\left(e^{-|\alpha|^2}e^{\alpha\hat{a}^\dagger}|0\rangle\langle 0|e^{\alpha^*\hat{a}}\right) = |\alpha\rangle\langle\alpha|\left(\hat{a} - \alpha\right)\end{aligned}$$

So,

$$
\begin{aligned}
\hat{a}|\alpha\rangle\langle\alpha|\hat{a}^\dagger &= \alpha|\alpha\rangle\langle\alpha|\alpha^* = |\alpha|^2|\alpha\rangle\langle\alpha| \\
\hat{a}^\dagger\hat{a}|\alpha\rangle\langle\alpha| &= \hat{a}^\dagger\alpha|\alpha\rangle\langle\alpha| = \alpha\left(\frac{\partial}{\partial\alpha}+\alpha^*\right)|\alpha\rangle\langle\alpha| \\
|\alpha\rangle\langle\alpha|\hat{a}^\dagger\hat{a} &= |\alpha\rangle\langle\alpha|\alpha^*\hat{a} = \alpha^*\left(\frac{\partial}{\partial\alpha^*}+\alpha\right)|\alpha\rangle\langle\alpha| \\
|\alpha\rangle\langle\alpha|\hat{a}\hat{a}^\dagger &= \left(\frac{\partial}{\partial\alpha^*}+\alpha\right)|\alpha\rangle\langle\alpha|\hat{a}^\dagger = \left(\frac{\partial}{\partial\alpha^*}+\alpha\right)\alpha^*|\alpha\rangle\langle\alpha| \\
\hat{a}^\dagger|\alpha\rangle\langle\alpha|\hat{a} &= \left(\frac{\partial}{\partial\alpha}+\alpha^*\right)|\alpha\rangle\langle\alpha|\hat{a} = \left(\frac{\partial}{\partial\alpha}+\alpha^*\right)\left(\frac{\partial}{\partial\alpha^*}+\alpha\right)|\alpha\rangle\langle\alpha|
\end{aligned}
$$

Using these results, one finds

$$
\begin{aligned}
&\int d^2\alpha\,|\alpha\rangle\langle\alpha|\frac{\partial}{\partial t}P(\alpha,t) \\
&= \int d^2\alpha\,P(\alpha,t)\left[-(\kappa+i\omega_0)\alpha\frac{\partial}{\partial\alpha}-(\kappa-i\omega_0)\alpha^*\frac{\partial}{\partial\alpha^*}+2\kappa\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]|\alpha\rangle\langle\alpha|
\end{aligned}
$$

The partial derivatives that act on $|\alpha\rangle\langle\alpha|$ can be transferred to the distribution $P(\alpha,t)$ by integrating by parts.

Assuming that $P(\alpha,t)$ vanishes sufficiently rapidly at infinity, we can drop the boundary terms to obtain

$$
\begin{aligned}
&\int d^2\alpha\,|\alpha\rangle\langle\alpha|\frac{\partial}{\partial t}P(\alpha,t) \\
&= \int d^2\alpha\,|\alpha\rangle\langle\alpha|\left[(\kappa+i\omega_0)\frac{\partial}{\partial\alpha}\alpha+(\kappa-i\omega_0)\frac{\partial}{\partial\alpha^*}\alpha^*+2\kappa\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P(\alpha,t)
\end{aligned}
$$

A sufficient condition for this equation to be satisfied is that the P distribution obeys the equation of motion

$$
\frac{\partial P}{\partial t} = \left[(\kappa+i\omega_0)\frac{\partial}{\partial\alpha}\alpha+(\kappa-i\omega_0)\frac{\partial}{\partial\alpha^*}\alpha^*+2\kappa\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P
$$

This is the *Fokker-Planck equation for the damped harmonic oscillator in the P representation*.

Note:
When taking derivatives with respect to complex variables, it is convenient to read the complex variable and its conjugate as two independent variables. This is allowed because

$$
\frac{\partial}{\partial\alpha}\alpha^* = \left(\frac{\partial}{\partial\alpha^*}\alpha\right)^* = \frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(x-iy) = \frac{1}{2}\left(\frac{\partial}{\partial x}x-\frac{\partial}{\partial y}y\right) = 0
$$

A similar approach is possible when integrating by parts. Explicitly, for given functions $f(\alpha)$ and $g(\alpha)$ (whose product vanishes at infinity), one can show that

$$
\begin{aligned}
\int d^2\alpha\,f(\alpha)\frac{\partial}{\partial\alpha}g(\alpha) &= -\int d^2\alpha\,g(\alpha)\frac{\partial}{\partial\alpha}f(\alpha) \\
\int d^2\alpha\,f(\alpha)\frac{\partial}{\partial\alpha^*}g(\alpha) &= -\int d^2\alpha\,g(\alpha)\frac{\partial}{\partial\alpha^*}f(\alpha)
\end{aligned}
$$

Properties of Fokker-Planck equations

A general Fokker-Planck Equation (FPE) in n variables may be written in the form

$$\frac{\partial}{\partial t}P(\mathbf{x},t) = \left[-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(\mathbf{x}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij}(\mathbf{x}) \right] P(\mathbf{x},t)$$

- The first derivative term determines the mean or deterministic motion and is called the *drift term*; $\mathbf{A} \equiv (A_i)$ is the *drift vector*.
- The second derivative term, provided its coefficient is positive definite, will cause a broadening or diffusion of $P(\mathbf{x},t)$ and is called the *diffusion term*; $\mathbf{D} \equiv (D_{ij})$ is the *diffusion matrix*.

Note: For a positive definite matrix $\mathbf{M}$, the quadratic form $\mathbf{z}^T\mathbf{M}\mathbf{z}$ is positive for all nontrivial $\mathbf{z}$.

The different role of the two terms may be seen in the equations of motion for $\langle x_k \rangle$ and $\langle x_k x_l \rangle$:

$$\frac{d}{dt}\langle x_k \rangle = \langle A_k \rangle\,, \qquad \frac{d}{dt}\langle x_k x_l \rangle = \langle x_k A_l \rangle + \langle x_l A_k \rangle + \frac{1}{2}\langle D_{kl} + D_{lk} \rangle$$

We see that A_k determines the motion of the mean amplitude whereas D_{lk} enters into the equation for the correlations.

Thus, from the FPE for the damped harmonic oscillator we have

$$\frac{d}{dt}\langle \alpha \rangle_P = -(\kappa + i\omega_0)\langle \alpha \rangle_P\,, \quad \frac{d}{dt}\langle \alpha^* \alpha \rangle_P = -2\kappa\langle \alpha^* \alpha \rangle_P + 2\kappa\bar{n}$$

which are equivalent to the equations of motion for $\langle \hat{a} \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle$ derived directly from the master equation.

Note that we define $\langle \alpha \rangle_P = \int d^2\alpha\, \alpha P(\alpha,t)$.

Solutions of the FPE

In general, finding solutions for $P(\alpha,t)$ analytically is impossible, but in certain situations steady state or even time-dependent solutions can be found.

Example: *Ornstein-Uhlenbeck process*

In the case where the drift term is linear in the variable $\mathbf{x}$ and the diffusion coefficient is a constant, i.e.,

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}(x_i P) + \frac{1}{2}\sum_{i,j=1}^{n} D_{ij}\frac{\partial^2 P}{\partial x_i \partial x_j}$$

a solution to the FPE may be found. in particular, for initial condition $P(\mathbf{x},0) = \delta^{(n)}(\mathbf{x} - \mathbf{x}^0)$ the solution is

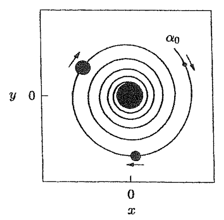
$$P(\mathbf{x},\mathbf{x}^0,t) = \frac{1}{\pi^{n/2}\{\det[\sigma(t)]\}^{1/2}} \exp\left\{ -\sum_{ij}[\sigma^{-1}(t)]_{ij}[x_i - x_i^0 e^{A_i t}][x_j - x_j^0 e^{A_j t}] \right\}$$

$$\text{with} \quad \sigma_{ij}(t) = \frac{-2D_{ij}}{A_i + A_j}\{1 - \exp[(A_i + A_j)t]\}$$

For a cavity mode coupled to a thermal reservoir and initially in a coherent state, i.e., $P(\alpha,0) = \delta^{(2)}(\alpha - \alpha_0)$, the solution is

$$P(\alpha,t) = \frac{1}{\pi\bar{n}(1 - e^{-2\kappa t})} \exp\left\{ -\frac{|\alpha - \alpha_0 e^{-(\kappa + i\omega_0)t}|^2}{\bar{n}(1 - e^{-2\kappa t})} \right\}$$

The coherent amplitude decays away and fluctuations from the reservoir cause its P function to assume a Gaussian form characteristic of thermal noise.



Notes

- From the above solution we may construct solutions for all initial conditions which have a non-singular P representation.
- It is not, however, possible to construct the solution for the oscillator initially in, e.g., a squeezed state, since no non-singular P function exists for such states.
- Alternative methods of converting the operator master equation to a c-number equation exist, based on the *Q and Wigner functions*, which can be used, e.g., for initial squeezed states.

Stochastic Differential Equations

- The FPE provides a dynamical description in terms of an evolving probability distribution which determines the average quantities that would be measured over an ensemble of experiments.
- An alternative approach to calculating these averages is to find a set of equations whose solutions generate trajectories in phase space, representative of a single experiment.
- Such trajectories must possess an irregular component modelling processes that are not observed in microscopic detail, but which manifest themselves macroscopically as sources of noise and fluctuations.
- These *stochastic trajectories* can be generated mathematically by stochastic differential equations – equations of motion that contain fluctuating source terms whose properties are defined probabilistically.

A FPE of the form

$$\frac{\partial}{\partial t}P(\mathbf{x},t)=\left[-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}A_i(\mathbf{x},t)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}D_{ij}(\mathbf{x},t)\right]P(\mathbf{x},t)$$

may be written in a completely equivalent form as the (Langevin) equation

$$\frac{d\mathbf{x}}{dt}=\mathbf{A}(\mathbf{x},t)+\mathbf{B}(\mathbf{x},t)\mathbf{E}(t)$$

where the matrix $\mathbf{B}(\mathbf{x},t)$ is defined by

$$\mathbf{B}(\mathbf{x},t)\mathbf{B}(\mathbf{x},t)^T=\mathbf{D}(\mathbf{x},t)$$

and $\mathbf{E}(t)$ are fluctuating forces with zero mean, i.e., $\langle E_i(t)\rangle=0$, and δ-correlated in time, i.e., $\langle E_i(t)E_j(t')\rangle=\delta_{ij}\delta(t-t')$.

Example:
Consider the *damped harmonic oscillator, coupled to a thermal reservoir*. The FPE is

$$\frac{\partial P}{\partial t}=\kappa\left(\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*+2\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)P$$

This describes an Ornstein-Uhlenbeck process (linear drift, constant diffusion). The diffusion matrix is

$$\mathbf{D}=2\kappa\bar{n}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

which may be factored as $\mathbf{D}=\mathbf{B}\mathbf{B}^T$, where

$$\mathbf{B}=\sqrt{\kappa\bar{n}}\begin{pmatrix}i&1\\-i&0\end{pmatrix}$$

Hence, the equivalent stochastic differential equations are

$$\frac{d}{dt}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} = \begin{pmatrix} -\kappa & 0 \\ 0 & -\kappa \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} + \sqrt{\kappa \bar{n}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix}$$

where $\eta_1(t)$ and $\eta_2(t)$ are independent stochastic "forces" which satisfy $\langle \eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta(t-t')$. These equations may be rewritten as

$$\frac{d\alpha}{dt} = -\kappa\alpha + \sqrt{2\kappa\bar{n}}\,\eta(t), \qquad \frac{d\alpha^*}{dt} = -\kappa\alpha^* + \sqrt{2\kappa\bar{n}}\,\eta^*(t)$$

where $\eta(t) = 2^{-1/2}[\eta_2(t) + i\eta_1(t)]$ is a complex stochastic force term satisfying $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta^*(t')\rangle = \delta(t-t')$.

The formal solution for $\alpha(t)$ is

$$\alpha(t) = \alpha(0)e^{-\kappa t} + \sqrt{2\kappa\bar{n}}\int_0^t ds\,\eta(s)e^{-\kappa(t-s)}$$

from which it follows that

$$\langle\alpha(t)\rangle = \langle\alpha(0)\rangle e^{-\kappa t}$$
$$\langle\alpha^*(t)\alpha(t)\rangle = \langle\alpha^*(0)\alpha(0)\rangle e^{-2\kappa t} + \bar{n}(1-e^{-2\kappa t})$$
$$\langle\alpha(t)\alpha(t)\rangle = \langle\alpha^*(t)\alpha^*(t)\rangle = 0$$

One can also show that

$$\langle\alpha^*(t)\alpha(t+\tau)\rangle_{ss} = \bar{n}e^{-\kappa\tau}$$

where $\tau \geq 0$ and 'ss' denotes the steady state.

Note:
For systems where a P representation exists the following results for *normally-ordered time correlation functions* may be proved:

$$G^{(1)}(t,\tau) = \langle\hat{a}^\dagger(t+\tau)\hat{a}(t)\rangle = \langle\alpha^*(t+\tau)\alpha(t)\rangle$$
$$G^{(2)}(t,\tau) = \langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\rangle = \langle|\alpha(t+\tau)|^2|\alpha(t)|^2\rangle$$

In these cases the measured correlation functions correspond to the same correlation function for the variables in the P representation. For non-normally-ordered correlation functions the result is not as simple.

Limitations

- The approaches outlined above (using P, Q, and Wigner representations) can provide a nice visualisation of quantum fluctuations in certain cases, but in general they are limited.
- In particular, the distributions may not satisfy a Fokker-Planck equation, or may require system-size expansions (i.e., small noise limits) in order to do so.
- This precludes them from being applied to systems, such as those encountered in cavity QED, where quantum fluctuations are large.
- Alternative approaches, i.e., *generalised P representations*,

$$\hat{\rho} = \int d^2\alpha \int d^2\alpha^\dagger \frac{|\alpha\rangle\langle\alpha^\dagger|}{\langle\alpha^\dagger|\alpha\rangle} P(\alpha,\alpha^\dagger) \quad \text{with} \quad (\alpha^\dagger)^* \neq \alpha^*$$

extend the phase space to accommodate large quantum noise, but can suffer from non-physical behaviour.

Theoretical Methods in Quantum Optics 5:
Master Equation Methods III

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29 September, 2008

Outline

Attempts to model quantum fluctuations using classical stochastic processes generally fail or encounter problems when these fluctuations are large. We now briefly outline an alternative approach, *quantum trajectories* (or *quantum Monte Carlo wave function simulations*), which provides a quantum stochastic process that is fully equivalent to the master equation and thereby enables the modelling and study of quantum optical systems exhibiting large quantum fluctuations.

Quantum Trajectories

- This approach is not founded upon a particular representation of the density operator.
- It sets up a *quantum stochastic process* that is fully equivalent to the master equation (plus the regression formula for correlation functions).
- It provides *visualisable realisations* (i.e., "trajectories") of *quantum fluctuations*.
- It has a natural connection with (and formulation in terms of) photoelectron counting measurements.

We aim to simulate a system described by the master equation

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[\hat{H}_S, \hat{\rho}] + \mathcal{L}\hat{\rho}$$

with

$$\mathcal{L}\hat{\rho} = -\frac{1}{2}\left(\hat{C}^\dagger\hat{C}\hat{\rho} + \hat{\rho}\hat{C}^\dagger\hat{C}\right) + \hat{C}^\dagger\hat{\rho}\hat{C}$$

where $\hat{C}$ is the system operator that appears in the coupling of the system to the reservoir (for example, $\hat{a}$).

We assume that at time t the system is in the state $|\psi(t)\rangle$. The evolution to the state at time $t+\delta t$ occurs in two steps.

- Firstly, assuming small δt, $|\psi_1(t+\delta t)\rangle$ is calculated according to

$$|\psi_1(t+\delta t)\rangle = \left(1 - \frac{i\hat{H}_{\rm eff}\delta t}{\hbar}\right)|\psi(t)\rangle$$

with the non-Hermitian effective Hamiltonian

$$\hat{H}_{\rm eff} = \hat{H}_S - \frac{1}{2}i\hbar\hat{C}^\dagger\hat{C}$$

Because $\hat{H}_{\rm eff}$ is non-Hermitian, $|\psi_1(t+\delta t)\rangle$ is not normalised, i.e.,

$$\langle\psi_1(t+\delta t)|\psi_1(t+\delta t)\rangle = 1 - \delta f$$

with

$$\delta f \simeq \delta t\,\frac{i}{\hbar}\langle\psi(t)|\hat{H}_{\rm eff} - \hat{H}_{\rm eff}^\dagger|\psi(t)\rangle = \delta t\,\langle\psi(t)|\hat{C}^\dagger\hat{C}|\psi(t)\rangle \ll 1$$

for small δt.

- Secondly, we test for the occurrence of a *quantum jump* (corresponding, e.g., to a photon emission/detection event).

To decide whether such a jump occurs we choose a random number, ϵ, from a uniform distribution on the interval [0,1].

- If $\delta f < \epsilon$ we deem no jump to occur and renormalise the state at time $t+\delta t$:

$$|\psi(t+\delta t)\rangle = \frac{|\psi_1(t+\delta t)\rangle}{\sqrt{1-\delta f}}$$

- If $\delta f > \epsilon$, we deem a jump to occur and set

$$|\psi(t+\delta t)\rangle = \frac{\hat{C}|\psi(t)\rangle}{\langle\psi(t)|\hat{C}^\dagger\hat{C}|\psi(t)\rangle} = \frac{\hat{C}|\psi(t)\rangle}{\sqrt{\delta f/\delta t}}$$

Averaging over the two possible outcomes for the density operator gives

$$\begin{aligned}\hat{\rho}(t+\delta t) &= (1-\delta f)\frac{|\psi_1(t+\delta t)\rangle}{\sqrt{1-\delta f}}\frac{\langle\psi_1(t+\delta t)|}{\sqrt{1-\delta f}} + \delta f\frac{\hat{C}|\psi(t)\rangle}{\sqrt{\delta f/\delta t}}\frac{\langle\psi(t)|\hat{C}^\dagger}{\sqrt{\delta f/\delta t}} \\ &= \hat{\rho}(t) - \delta t\frac{i}{\hbar}[\hat{H}_S,\hat{\rho}(t)] + \delta t\,\mathcal{L}(\hat{\rho}(t))\end{aligned}$$

and taking the limit $\delta t \to 0$ we find

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_S,\hat{\rho}] + \mathcal{L}\hat{\rho}$$

which is just the master equation.

In the case where the initial state is not a pure state, one has first to decompose it as a statistical mixture of pure states, $\hat{\rho}(0) = \sum p_i|\chi_i\rangle\langle\chi_i|$, and then randomly choose the initial wave function among the $\{|\chi_i\rangle\}$ according to the probability distribution $\{p_i\}$.

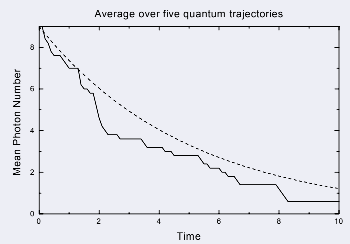
Damped cavity mode in initial Fock state $|n=9\rangle$

For a damped cavity mode we have

$$\hat{H}_S = \hbar\omega\hat{a}^\dagger\hat{a} \quad \text{and} \quad \hat{C} = \sqrt{2\kappa}\,\hat{a}$$

Given an initial state $\hat{\rho}(0) = |9\rangle\langle 9|$, the mean photon number in the mode is given by (dashed line)

$$\langle\hat{n}(t)\rangle = e^{-2\kappa t}\langle\hat{n}(0)\rangle = 9e^{-2\kappa t}$$



Theoretical Methods in Quantum Optics 6:
Input-Output Formulation of Optical Cavities

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29 September, 2008

Outline

The master equation provides a means of computing the photon statistics inside an optical cavity, but it is the field external to the cavity that is ultimately measured. By treating the dynamics of the external field explicitly (rather than eliminating it in the role of a passive heat bath), one can derive relationships between the input, output, and intracavity fields.

Topics

- Cavity Modes
- Linear Systems
- Two-Time Correlation Functions
- Spectrum of Squeezing
- Parametric Amplifier

Cavity Modes

We consider a single optical cavity mode coupled to an external, one-dimensional (multimode) field. The total Hamiltonian is

$$\hat{H} = \hat{H}_{\rm sys} + \hat{H}_{\rm res} + \hat{H}_{\rm int}$$

where $\hat{H}_{\rm sys}$ is the free Hamiltonian for the intracavity field mode, $\hat{H}_{\rm res}$ is the free Hamiltonian for the external (or reservoir) field modes, and

$$\hat{H}_{\rm int} = i\hbar \int_{-\infty}^{\infty} d\omega\, \kappa(\omega) \left[\hat{a}^\dagger \hat{b}(\omega) - \hat{b}^\dagger(\omega)\hat{a} \right]$$

with $\hat{a}$ and $\hat{b}(\omega)$ annihilation operators for the intracavity and external field, respectively, satisfying commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{b}(\omega), \hat{b}^\dagger(\omega')] = \delta(\omega - \omega')$$

and $\kappa(\omega)$ a coupling constant.

Note:
The actual physical frequency limits in the integral are $(0, \infty)$. However, for high frequencies we may shift the integration to a frequency Ω characteristic of the system (e.g., the cavity resonance frequency), and the integration limits become $(-\Omega, \infty)$. As Ω is large, extending the lower limit to $-\infty$ is a good approximation.

The Heisenberg equation of motion for $\hat{b}(\omega)$ is

$$\dot{\hat{b}}(\omega) = -\mathrm{i}\omega \hat{b}(\omega) + \kappa(\omega)\hat{a}$$

A formal solution may be written in terms of *initial* (t_0) or *final* (t_1) conditions (i.e., *input* or *output*):

$$\begin{aligned} \hat{b}(\omega, t) &= e^{-\mathrm{i}\omega(t-t_0)}\hat{b}(\omega, t_0) + \kappa(\omega)\int_{t_0}^{t} \mathrm{d}t'\ e^{-\mathrm{i}\omega(t-t')}\hat{a}(t')\,, \quad t_0 < t \\ &= e^{-\mathrm{i}\omega(t-t_1)}\hat{b}(\omega, t_1) - \kappa(\omega)\int_{t}^{t_1} \mathrm{d}t'\ e^{-\mathrm{i}\omega(t-t')}\hat{a}(t')\,, \quad t < t_1 \end{aligned}$$

We can substitute one of these solutions for $\hat{b}(\omega, t)$ into the equation of motion for the system operator $\hat{a}$, i.e.,

$$\begin{aligned} \dot{\hat{a}}(t) &= -\frac{\mathrm{i}}{\hbar}[\hat{a}(t), \hat{H}_{\mathrm{sys}}] - \int_{-\infty}^{\infty} \mathrm{d}\omega\ \kappa(\omega)\hat{b}(\omega, t) \\ &= -\frac{\mathrm{i}}{\hbar}[\hat{a}(t), \hat{H}_{\mathrm{sys}}] - \int_{-\infty}^{\infty} \mathrm{d}\omega\ \kappa(\omega)e^{-\mathrm{i}\omega(t-t_0)}\hat{b}(\omega, t_0) \\ &\quad - \int_{-\infty}^{\infty} \mathrm{d}\omega\ \kappa(\omega)^2 \int_{t_0}^{t} \mathrm{d}t'\ e^{-\mathrm{i}\omega(t-t')}\hat{a}(t') \end{aligned}$$

We now assume that $\kappa(\omega)$ is independent of frequency over a band of frequencies about the cavity mode frequency, i.e., we set

$$\kappa(\omega)^2 = \kappa/\pi$$

Then, using $\int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega(t-t')} = 2\pi\delta(t-t')$, we can derive

$$\dot{\hat{a}}(t) = -(\mathrm{i}/\hbar)[\hat{a}(t), \hat{H}_{\mathrm{sys}}] - \kappa\hat{a}(t) + \sqrt{2\kappa}\,\hat{a}_{\mathrm{in}}(t)$$

where we define the *input field operator*

$$\hat{a}_{\mathrm{in}}(t) \equiv \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\ e^{-\mathrm{i}\omega(t-t_0)}\hat{b}(\omega, t_0)$$

which satisfies $[\hat{a}_{\mathrm{in}}(t), \hat{a}_{\mathrm{in}}^{\dagger}(t')] = \delta(t-t')$.

This is a *quantum Langevin equation* for the damped amplitude $\hat{a}(t)$ in which the (quantum) noise term appears explicitly as the input field.

We can also substitute for $\hat{b}(\omega, t)$ in terms of the output field (time t_1), which leads to

$$\dot{\hat{a}}(t) = -(\mathrm{i}/\hbar)[\hat{a}(t), \hat{H}_{\mathrm{sys}}] + \kappa\hat{a}(t) - \sqrt{2\kappa}\,\hat{a}_{\mathrm{out}}(t)$$

with the *output field operator* defined by

$$\hat{a}_{\mathrm{out}}(t) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\ e^{-\mathrm{i}\omega(t-t_1)}\hat{b}(\omega, t_1)$$

which satisfies $[\hat{a}_{\mathrm{out}}(t), \hat{a}_{\mathrm{out}}^{\dagger}(t')] = \delta(t-t')$.

Input-output relation

A relation between the external fields and the intracavity field may be obtained by equating the two expressions for $\dot{\hat{a}}(t)$, which gives

$$\hat{a}_{\text{out}}(t) + \hat{a}_{\text{in}}(t) = \sqrt{2\kappa}\,\hat{a}(t)$$

This is a boundary condition relating each of the far-field amplitudes outside the cavity to the internal cavity field.

Note:
It is important to note that "interference" terms like, e.g., $\langle a(t) a_{\text{in}}(t') \rangle$ and $\langle a^\dagger(t) a_{\text{in}}(t') \rangle$, may contribute to observed output field moments.

Linear Systems

For many systems of interest, the Heisenberg equations may be linear:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{a}}(t) = \mathbf{A}\hat{\mathbf{a}}(t) - \kappa\hat{\mathbf{a}}(t) + \sqrt{2\kappa}\,\hat{\mathbf{a}}_{\text{in}}(t)$$

where

$$\hat{\mathbf{a}}(t) = \begin{bmatrix} \hat{a}(t) \\ \hat{a}^\dagger(t) \end{bmatrix}, \qquad \hat{\mathbf{a}}_{\text{in}}(t) = \begin{bmatrix} \hat{a}_{\text{in}}(t) \\ \hat{a}_{\text{in}}^\dagger(t) \end{bmatrix}$$

Define the Fourier transform

$$\hat{a}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega\; e^{-\mathrm{i}\omega(t-t_0)}\hat{a}(\omega) \qquad \text{and} \qquad \hat{\mathbf{a}}(\omega) = \begin{bmatrix} \hat{a}(\omega) \\ \hat{a}^\dagger(\omega) \end{bmatrix}$$

where $\hat{a}^\dagger(\omega)$ is the Fourier transform of $\hat{a}^\dagger(t)$.

In the Fourier-transformed space, the equations of motion become

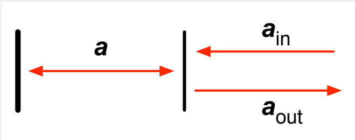
$$[\mathbf{A} + (\mathrm{i}\omega - \kappa)\,\mathbf{I}]\,\hat{\mathbf{a}}(\omega) = -\sqrt{2\kappa}\hat{\mathbf{a}}_{\text{in}}(\omega)$$

where $\mathbf{I}$ is the identity matrix. Using the input-output relation to eliminate the internal mode, we obtain

$$\hat{\mathbf{a}}_{\text{out}}(\omega) = -\,[\mathbf{A} + (\mathrm{i}\omega + \kappa)\,\mathbf{I}]\,[\mathbf{A} + (\mathrm{i}\omega - \kappa)\,\mathbf{I}]^{-1}\,\hat{\mathbf{a}}_{\text{in}}(\omega)$$

Example: One-sided cavity

The only source of loss in the cavity is through the mirror which couples the input and output fields.



$$\hat{H}_{\text{sys}} = \hbar\omega_0\hat{a}^\dagger\hat{a} \qquad \text{so} \qquad \mathbf{A} = \begin{pmatrix} -\mathrm{i}\omega_0 & 0 \\ 0 & \mathrm{i}\omega_0 \end{pmatrix}$$

$$\text{and} \qquad \hat{\mathbf{a}}_{\text{out}}(\omega) = \frac{\kappa + \mathrm{i}(\omega-\omega_0)}{\kappa - \mathrm{i}(\omega-\omega_0)}\,\hat{\mathbf{a}}_{\text{in}}(\omega)$$

Hence, there is a frequency dependent phase shift between the output and input.

Two-Time Correlation Functions

If we integrate $\hat{b}(\omega, t)$ over frequency we obtain

$$\hat{a}_{\text{in}}(t) = \sqrt{\kappa/2}\,\hat{a}(t) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \hat{b}(\omega, t)$$

Let $\hat{c}(t)$ be any system operator. Then

$$[\hat{c}(t), \sqrt{2\kappa}\,\hat{a}_{\text{in}}(t)] = \kappa\,[\hat{c}(t), \hat{a}(t)]$$

since $[\hat{c}(t), \hat{b}(\omega, t)] = 0$. Now, since $\hat{c}(t)$ can only be a function of $\hat{a}_{\text{in}}(t')$ for earlier times $t' < t$ (due to causality), and the input field operators must commute at different times, we have

$$[\hat{c}(t), \sqrt{2\kappa}\,\hat{a}_{\text{in}}(t')] = 0, \quad t' > t$$

Similarly,

$$[\hat{c}(t), \sqrt{2\kappa}\,\hat{a}_{\text{out}}(t')] = 0, \quad t' < t$$

Using this result and the input-output relation, we then have

$$[\hat{c}(t), \sqrt{2\kappa}\,\hat{a}_{\text{in}}(t')] = 2\kappa\,[\hat{c}(t), \hat{a}(t')], \quad t' < t$$

or, in general,

$$[\hat{c}(t), \sqrt{2\kappa}\,\hat{a}_{\text{in}}(t')] = 2\kappa\theta(t - t')[\hat{c}(t), \hat{a}(t')]$$

where $\theta(t)$ is the step function:

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$

- For *coherent* or *vacuum* inputs to the cavity, it is now possible to express correlation functions of the output field entirely in terms of those of the internal mode.
- In particular, for inputs of this sort, moments of the form $\langle \hat{a}_{\text{in}}^\dagger(t)\hat{a}_{\text{in}}(t')\rangle$, $\langle \hat{a}(t)\hat{a}_{\text{in}}(t')\rangle$, $\langle \hat{a}^\dagger(t)\hat{a}_{\text{in}}(t')\rangle$, and $\langle \hat{a}_{\text{in}}^\dagger(t)\hat{a}^\dagger(t')\rangle$ factorise, and, defining $\langle u, v\rangle \equiv \langle uv\rangle - \langle u\rangle\langle v\rangle$, we find

$$\langle \hat{a}_{\text{out}}^\dagger(t), \hat{a}_{\text{out}}(t')\rangle = 2\kappa\langle \hat{a}^\dagger(t), \hat{a}(t')\rangle$$

and

$$\langle \hat{a}_{\text{out}}(t), \hat{a}_{\text{out}}(t')\rangle = 2\kappa\langle \hat{a}(\max[t, t']), \hat{a}(\min[t, t'])\rangle$$

Spectrum of Squeezing

The output field from a cavity is a continuum of frequencies. One defines the intensity spectrum of this field as the Fourier transform of the phase-independent correlation function $\langle \hat{a}_{\text{out}}^\dagger(t), \hat{a}_{\text{out}}(t')\rangle$.

Similarly, the squeezing spectrum can be defined as the Fourier transform of an appropriate phase-dependent correlation function, and it gives the squeezing in the frequency components of an appropriate quadrature phase operator.

We define the output field quadrature phase operators as

$$\begin{aligned} \hat{X}_1^{\text{out}}(t) &= \hat{a}_{\text{out}}(t)e^{-i(\theta-\Omega t)} + \hat{a}_{\text{out}}^\dagger(t)e^{i(\theta-\Omega t)} \\ \hat{X}_2^{\text{out}}(t) &= -i\left[\hat{a}_{\text{out}}(t)e^{-i(\theta-\Omega t)} - \hat{a}_{\text{out}}^\dagger(t)e^{i(\theta-\Omega t)}\right] \end{aligned}$$

where Ω is the reference frequency (typically the cavity frequency) and θ the reference phase.

The squeezing spectrum is defined as the Fourier transform of the normally-ordered two-time correlation function $\langle : \hat{X}_i^{\rm out}(t), \hat{X}_i^{\rm out}(0) : \rangle$,

$$
\begin{aligned}
: S_i^{\rm out}(\omega) : &= \int {\rm d}t \, \langle : \hat{X}_i^{\rm out}(t), \hat{X}_i^{\rm out}(0) : \rangle \, e^{-{\rm i}\omega t} \\
&= 2\kappa \int {\rm d}t \, \mathcal{T} \langle : \hat{X}_i(t), \hat{X}_i(0) : \rangle \, e^{-{\rm i}\omega t}
\end{aligned}
$$

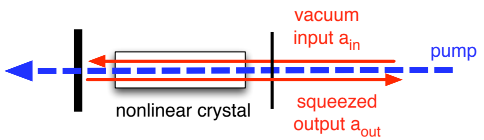
where $\mathcal{T}$ denotes time ordering and we have used the input-output relations to express the output correlation function in terms of the intracavity quadrature phase operators,

$$
\hat{X}_1(t) = \hat{a}(t)e^{-{\rm i}\theta} + \hat{a}^\dagger(t)e^{{\rm i}\theta}, \quad \hat{X}_2(t) = -{\rm i}[\hat{a}(t)e^{-{\rm i}\theta} - \hat{a}^\dagger(t)e^{{\rm i}\theta}]
$$

where $\hat{a}(t), \hat{a}^\dagger(t)$ are defined in a frame rotating at frequency Ω.

Parametric Amplifier

We now compute the squeezing spectrum from the output of a parametric amplifier.



Treating the pump field (of frequency $2\omega_0$) classically, we can write

$$
\hat{H}_{\rm sys} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + ({\rm i}\hbar/2)\left[\epsilon e^{-2{\rm i}\omega_0 t}(\hat{a}^\dagger)^2 - \epsilon^* e^{2{\rm i}\omega_0 t}\hat{a}^2\right]
$$

where $\epsilon = |\epsilon| e^{{\rm i}\theta}$.

The Heisenberg equations of motion for $\hat{\mathbf{a}}$ are linear, and, in a frame rotating at frequency ω_0, the matrix $\mathbf{A}$ is given by

$$
\mathbf{A} = \begin{bmatrix} \kappa & -\epsilon \\ -\epsilon^* & \kappa \end{bmatrix}
$$

The Fourier components of the output field are found to be

$$
\hat{a}_{\rm out}(\omega) = \frac{1}{(\kappa - {\rm i}\omega)^2 - |\epsilon|^2} \left\{ \left(\kappa^2 + \omega^2 + |\epsilon|^2\right) \hat{a}_{\rm in}(\omega) + 2\epsilon\kappa \hat{a}_{\rm in}^\dagger(-\omega) \right\}
$$

Defining the quadrature operators in this case by $\hat{a}_{\rm out}(t) = (1/2)e^{{\rm i}\theta/2}[\hat{X}_1^{\rm out}(t) + {\rm i}\hat{X}_2^{\rm out}(t)]$, the solution for $\hat{a}_{\rm out}(\omega)$ can be used directly to give the squeezing spectra (remember that $\omega = 0$ corresponds to the cavity resonance):

$$
\begin{aligned}
S_1^{\rm out}(\omega) &= 1 + : S_1^{\rm out}(\omega) : = 1 + \frac{4\kappa|\epsilon|}{(\kappa - |\epsilon|)^2 + \omega^2} \\
S_2^{\rm out}(\omega) &= 1 + : S_2^{\rm out}(\omega) : = 1 - \frac{4\kappa|\epsilon|}{(\kappa + |\epsilon|)^2 + \omega^2}
\end{aligned}
$$

- So, squeezing $[S_i^{\rm out}(\omega) < 1]$ occurs in the $\hat{X}_2$ quadrature.
- Perfect squeezing, $S_2^{\rm out}(\omega) \to 0$, occurs at $\omega = 0$ in the limit $|\epsilon| \to \kappa$.

Theoretical Methods in Quantum Optics 7:
Interaction of Radiation with Atoms

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29 September, 2008

Outline

The interaction between the quantised EM field and an atom represents one of the most fundamental problems in quantum optics. Real atoms have complicated energy level structures, but, in many instances, only two atomic energy levels play a significant role in the interaction with the EM field (due, e.g., to selection rules). So, it is common in theoretical treatments to represent the atom by a quantum system with only two energy eigenstates. Here we outline the derivation of such models and consider some elementary, but fundamentally interesting, properties and phenomena.

Topics

- Two-State Atoms
- Atom-Field Interaction
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- Resonance Fluorescence
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Two-State Atoms

We consider an atom with two states, $|1\rangle$ and $|2\rangle$, having energies E_1 and E_2 with $E_1 < E_2$, between which radiative transitions are allowed. Adopting these energy eigenstates as a basis for our two-level atom, the unperturbed atomic Hamiltonian $\hat{H}_A$ can be written in the form

$$\begin{aligned} \hat{H}_A &= E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| \\ &= \frac{1}{2}(E_1+E_2)\hat{I} + \frac{1}{2}(E_2-E_1)\hat{\sigma}_z \end{aligned}$$

where $\hat{\sigma}_z \equiv |2\rangle\langle 2| - |1\rangle\langle 1|$, and $\hat{I} \equiv |1\rangle\langle 1| + |2\rangle\langle 2|$ is the identity. The first term in $\hat{H}_A$ is a constant which may be eliminated by referring the atomic energies to the middle of the atomic transition. We then write

$$\hat{H}_A = \frac{1}{2}\hbar\omega_A\hat{\sigma}_z\,, \qquad \omega_A \equiv (E_2-E_1)/\hbar$$

Consider now the dipole moment operator $e\hat{\mathbf{r}}$, where e is the electronic charge and $\hat{\mathbf{r}}$ is the coordinate operator for the bound electron:

$$\begin{aligned} e\hat{\mathbf{r}} &= e\sum_{n,m=1}^{2}\langle n|\hat{\mathbf{r}}|m\rangle\,|n\rangle\langle m| \\ &= e\left(\langle 1|\hat{\mathbf{r}}|2\rangle\,|1\rangle\langle 2| + \langle 2|\hat{\mathbf{r}}|1\rangle\,|2\rangle\langle 1|\right) = \mathbf{d}_{12}\hat{\sigma}_- + \mathbf{d}_{21}\hat{\sigma}_+ \end{aligned}$$

where we have set $\langle 1|\hat{\mathbf{r}}|1\rangle = \langle 2|\hat{\mathbf{r}}|2\rangle = 0$ (assuming atomic states whose symmetry guarantees zero permanent dipole moment), and we have introduced the *atomic dipole matrix elements*

$$\mathbf{d}_{12} \equiv e\langle 1|\hat{\mathbf{r}}|2\rangle = e\int d^3r\,\phi_2^*(\mathbf{r})\mathbf{r}\phi_1(\mathbf{r}), \qquad \mathbf{d}_{21} = (\mathbf{d}_{12})^*$$

with $\phi_i(\mathbf{r})$ the (unperturbed) electron wave functions. We have also introduced the *atomic lowering and raising operators*

$$\hat{\sigma}_- \equiv |1\rangle\langle 2|, \qquad \hat{\sigma}_+ \equiv |2\rangle\langle 1|$$

The matrix representations for the operators $\hat{\sigma}_z$, $\hat{\sigma}_-$ and $\hat{\sigma}_+$ are

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We may also identify $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm i\hat{\sigma}_y)$, where

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The matrices $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are the *Pauli spin matrices* introduced initially in the context of magnetic transitions in spin-1/2 systems.

Properties of the spin operators

It is straightforward to show that

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z\,, \qquad [\hat{\sigma}_\pm, \hat{\sigma}_z] = \mp 2\hat{\sigma}_\pm\,, \qquad \hat{\sigma}_+\hat{\sigma}_- + \hat{\sigma}_-\hat{\sigma}_+ = \hat{I}$$

$$\begin{aligned} \hat{\sigma}_z|1\rangle &= -|1\rangle, \qquad \hat{\sigma}_z|2\rangle = |2\rangle \\ \hat{\sigma}_-|1\rangle &= 0, \qquad \hat{\sigma}_-|2\rangle = |1\rangle \\ \hat{\sigma}_+|1\rangle &= |2\rangle, \qquad \hat{\sigma}_+|2\rangle = 0 \end{aligned}$$

For an *atomic state specified by a density operator* $\hat{\rho}$, expectation values of $\hat{\sigma}_z$, $\hat{\sigma}_-$ and $\hat{\sigma}_+$ are just matrix elements of the density operator, and give the *population difference* (or *inversion*)

$$\langle\hat{\sigma}_z\rangle = \mathrm{Tr}(\hat{\rho}\hat{\sigma}_z) = \langle 2|\hat{\rho}|2\rangle - \langle 1|\hat{\rho}|1\rangle = \rho_{22} - \rho_{11},$$

and the mean *atomic polarisation*

$$\langle e\hat{\mathbf{r}}\rangle = \mathbf{d}_{12}\mathrm{Tr}(\hat{\rho}\hat{\sigma}_-) + \mathbf{d}_{21}\mathrm{Tr}(\hat{\rho}\hat{\sigma}_+) = \mathbf{d}_{12}\rho_{21} + \mathbf{d}_{21}\rho_{12}$$

Atom-Field Interaction

Consider a two-level atom coupled to the EM field, represented as usual by a collection of quantised harmonic oscillators. Within the rotating-wave and dipole approximations, we write

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{AF}$$

where

$$\begin{aligned} \hat{H}_A &= \frac{1}{2}\hbar\omega_A\hat{\sigma}_z, \qquad \hat{H}_F = \sum_{\mathbf{k},\lambda} \hbar\omega_k \hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} \\ \hat{H}_{AF} &= \sum_{\mathbf{k},\lambda} \hbar\left(\kappa^*_{\mathbf{k}\lambda}\hat{a}^\dagger_{\mathbf{k}\lambda}\hat{\sigma}_- + \kappa_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda}\hat{\sigma}_+\right) \end{aligned}$$

with

$$\kappa_{\mathbf{k}\lambda} = -i\sqrt{\frac{\omega_k}{2\hbar\epsilon_0}}\,\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}_A)\cdot\mathbf{d}_{21}$$

Notes:

- In the *dipole approximation* the field is assumed to be uniform over the extent of the atom. In the optical regime this is valid because the wavelength of light $\sim 10^2$nm $\gg r_{\mathrm{atom}} \sim 0.1$ nm.
- The summation extends over field modes with wavevectors $\mathbf{k}$ and polarisation states λ (and corresponding frequencies ω_k).
- The atom is positioned at $\mathbf{r}_A$, and $\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}_A)$ is a field mode function at that point. In *free space*, for example,

  $$\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}_A) = \frac{1}{\sqrt{V}}\tilde{\mathbf{e}}_{\mathbf{k}\lambda}e^{i\mathbf{k}\cdot\mathbf{r}_A}$$

  where $\tilde{\mathbf{e}}_{\mathbf{k}\lambda}$ is the unit polarisation vector and V the quantisation volume.
- The interaction Hamiltonian $\hat{H}_{AF}$ follows from the familiar expression $-e\hat{\mathbf{r}}\cdot\hat{\mathbf{E}}(\mathbf{r}_A)$ for the potential energy of a dipole in a field.

Spontaneous Decay of a Two-Level Atom

The master equation for the reduced density operator $\hat{\rho}$ of a radiatively damped two-level atom in free space is derived as

$$\begin{aligned}\dot{\hat{\rho}} &= -i\frac{1}{2}\omega_A'[\hat{\sigma}_z, \hat{\rho}] + \frac{1}{2}\gamma(\bar{n}+1)\left(2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-\right) \\ &\quad + \frac{1}{2}\gamma\bar{n}\left(2\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \hat{\rho}\hat{\sigma}_-\hat{\sigma}_+\right)\end{aligned}$$

where $\omega_A' = \omega_A + 2\Delta' + \Delta$, $\bar{n} = \bar{n}(\omega_A, T)$, and, in integral form,

$$\begin{aligned}\gamma &= 2\pi\sum_\lambda\int d^3k\, g(\mathbf{k})|\kappa(\mathbf{k},\lambda)|^2\delta(kc-\omega_A) \\ \Delta &= \sum_\lambda P\int d^3k\, \frac{g(\mathbf{k})|\kappa(\mathbf{k},\lambda)|^2}{\omega_A - kc} \\ \Delta' &= \sum_\lambda P\int d^3k\, \frac{g(\mathbf{k})|\kappa(\mathbf{k},\lambda)|^2}{\omega_A - kc}\,\bar{n}(kc, T)\end{aligned}$$

Notes:

- The factor $(\gamma/2)(\bar{n}+1)$ contains a rate for *spontaneous transitions*, independent of $\bar{n}$, and a rate for *stimulated transitions* induced by thermal photons, proportional to $\bar{n}$.
- The factor $(\gamma/2)\bar{n}$ gives a rate for *absorptive transitions* which take thermal photons from the equilibrium EM field.
- The quantity $\omega_A' - \omega_A = 2\Delta' + \Delta$ describes the *Lamb shift*, including a temperature-dependent contribution $2\Delta'$ that does not appear for the harmonic oscillator. Its appearance here is a consequence of the commutator $[\hat{\sigma}_-, \hat{\sigma}_+] = -\hat{\sigma}_z$, in place of the corresponding $[\hat{a}, \hat{a}^\dagger] = 1$.
- Note, however, that the rotating-wave approximation we have adopted does not in fact give the correct nonrelativistic result for the Lamb shift. Actually, $(\omega_A - kc)^{-1}$ should be replaced with $(\omega_A - kc)^{-1} + (\omega_A + kc)^{-1}$.

The Einstein A coefficient

By performing the integration over wavevectors and summing over the polarisations, one can show that

$$\gamma = \frac{1}{4\pi\epsilon_0}\frac{4\omega_A^3 d_{12}^2}{3\hbar c^3}$$

which is the Einstein A coefficient (as it must be).

Matrix element equations

From the master equation, we derive (using $\langle\dot{\hat{\sigma}}_i\rangle = \mathrm{Tr}(\hat{\sigma}_i\dot{\hat{\rho}})$ and the properties of the spin operators)

$$\begin{aligned}\langle\dot{\hat{\sigma}}_z\rangle &= -\gamma\left[\langle\hat{\sigma}_z\rangle(2\bar{n}+1)+1\right] \\ \langle\dot{\hat{\sigma}}_-\rangle &= -\left[\frac{1}{2}\gamma(2\bar{n}+1) + i\omega_A\right]\langle\hat{\sigma}_-\rangle \\ \langle\dot{\hat{\sigma}}_+\rangle &= -\left[\frac{1}{2}\gamma(2\bar{n}+1) - i\omega_A\right]\langle\hat{\sigma}_+\rangle\end{aligned}$$

Notes:

- We drop the distinction between ω_A and ω_A'.
- At optical frequencies and normal laboratory temperatures $\bar{n}$ is negligible, so for simplicity we set $\bar{n} = 0$ from now on.

Correlation functions

To compute correlation functions we use the quantum regression formula. Noting that $\hat{\sigma}_+\hat{\sigma}_- = (1/2)(1+\hat{\sigma}_z)$, we may write the mean-value equations in vector form:

$$\langle \dot{\mathbf{s}} \rangle = \mathbf{M}\langle \mathbf{s} \rangle$$

with

$$\mathbf{s} \equiv \begin{pmatrix} \hat{\sigma}_- \\ \hat{\sigma}_+ \\ \hat{\sigma}_+\hat{\sigma}_- \end{pmatrix} \qquad \mathbf{M} \equiv \begin{pmatrix} -\frac{1}{2}\gamma + i\omega_A & 0 & 0 \\ 0 & -\frac{1}{2}\gamma + i\omega_A & 0 \\ 0 & 0 & -\gamma \end{pmatrix}$$

From the quantum regression theorem it follows that, for example,

$$\frac{d}{d\tau}\langle \hat{\sigma}_+(t)\mathbf{s}(t+\tau)\rangle \;=\; \mathbf{M}\langle \hat{\sigma}_+(t)\mathbf{s}(t+\tau)\rangle$$

Spontaneous emission spectrum for an initially excited atom

The spectrum is defined in terms of the probability for photodetection by a monochromatic detector a distance r from the source during an interval T. For an optical frequency field and an ideal detector, the spectrum is given by

$$S(\omega,\mathbf{r},T) = \frac{1}{2\pi}\int_{r/c}^{T+r/c} dt_1 \int_{r/c}^{T+r/c} dt_2 \; e^{i\omega(t_2-t_1)} G^{(1)}(\mathbf{r},t_1;\mathbf{r},t_2)$$

$$\text{where} \quad G^{(1)}(\mathbf{r},t_1;\mathbf{r},t_2) = \langle \hat{\mathbf{E}}^{(-)}_{\text{out}}(\mathbf{r},t_1)\cdot\hat{\mathbf{E}}^{(+)}_{\text{out}}(\mathbf{r},t_2)\rangle$$

with

$$\hat{\mathbf{E}}^{(+)}_{\text{out}}(\mathbf{r},t) = \hat{\mathbf{E}}^{(+)}_{\text{in}}(\mathbf{r},t) - \frac{\omega_A^2}{4\pi\epsilon_0 c^2 r}\left[\left(\mathbf{d}_{12}\times\frac{\mathbf{r}}{r}\right)\times\frac{\mathbf{r}}{r}\right]\hat{\sigma}_-(t-r/c)$$

This is the retarded field generated by a point dipole with the classical dipole moment replaced by the atomic lowering operator $\hat{\sigma}_-$.

Using this, we can derive

$$G^{(1)}(\mathbf{r},t_1;\mathbf{r},t_2) = I_0(\mathbf{r})\langle \hat{\sigma}_+(\tilde{t}_1)\hat{\sigma}_-(\tilde{t}_2)\rangle$$

where $\tilde{t} = t - (r/c)$ and $I_0(\mathbf{r})$ is a geometrical factor given by

$$I_0(\mathbf{r}) = \left|\frac{\omega_A^2}{4\pi\epsilon_0 c^2 r}\left(\mathbf{d}_{12}\times\frac{\mathbf{r}}{r}\right)\times\frac{\mathbf{r}}{r}\right|^2$$

Neglecting r/c compared to t and T, and taking the limit $T\to\infty$ (i.e., counting time long compared to the spontaneous emission lifetime γ^{-1}), the spectrum follows as

$$S(\omega,\mathbf{r},\infty) = \frac{I_0(\mathbf{r})}{2\pi}\frac{1}{(\omega-\omega_A)^2 + (\gamma/2)^2}$$

This is the familiar *Lorentzian lineshape* of the Wigner-Weisskopf theory, with halfwidth equal to $\gamma/2$.

Resonance Fluorescence

We now consider a two-level atom irradiated by a strong monochromatic laser beam tuned to the atomic transition. Photons may be absorbed from this beam and emitted to the many modes of the vacuum electromagnetic field as fluorescent scattering.

As we will see, a two-level atom responds nonlinearly to increasing laser intensity. The fluorescence spectrum acquires an incoherent component having the natural linewidth γ. This incoherent spectrum splits into a three-peaked structure (the *Mollow triplet*) and eventually accounts for nearly all of the scattered intensity. The incoherent spectral component arises from quantum fluctuations around the nonequilibrium steady state established by the balance between excitation and emission processes.

Master equation for resonance fluorescence

The incident laser mode is in a highly excited state that is essentially unaffected by its interaction with the single atom, so we can treat this field as a classical driving force. The master equation is then

$$\begin{aligned}\dot{\hat{\rho}} &= -i\frac{1}{2}\omega_A[\hat{\sigma}_z, \hat{\rho}] + i(\Omega/2)[e^{-i\omega_A t}\hat{\sigma}_+ + e^{i\omega_A t}\hat{\sigma}_-, \hat{\rho}] \\ &\quad + \frac{1}{2}\gamma\left(2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_- - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho}\right)\end{aligned}$$

where $\Omega \equiv 2\left(\frac{dE}{\hbar}\right)$ is the *Rabi frequency*.

Note:
The laser field at the site of the atom is $\mathbf{E}(t) = \tilde{\mathbf{e}}\,2E\cos(\omega_A t + \phi)$, where $\tilde{\mathbf{e}}$ is a unit polarisation vector, E is a real amplitude, and the phase ϕ is chosen so that $d \equiv \tilde{\mathbf{e}} \cdot \mathbf{d}_{12} e^{i\phi}$ is also real.

Optical Bloch equations

From the master equation we obtain the *optical Bloch equations* with radiative damping (so called for their relationship to the equations of a spin-1/2 particle in a magnetic field), which, in a frame rotating at frequency ω_A, take the form

$$\begin{aligned}\langle\dot{\tilde{\sigma}}_-\rangle &= -i(\Omega/2)\langle\tilde{\sigma}_z\rangle - \frac{1}{2}\gamma\langle\tilde{\sigma}_-\rangle \\ \langle\dot{\tilde{\sigma}}_+\rangle &= i(\Omega/2)\langle\tilde{\sigma}_z\rangle - \frac{1}{2}\gamma\langle\tilde{\sigma}_+\rangle \\ \langle\dot{\tilde{\sigma}}_z\rangle &= i\Omega\langle\tilde{\sigma}_+\rangle - i\Omega\langle\tilde{\sigma}_-\rangle - \gamma(\langle\tilde{\sigma}_z\rangle + 1)\end{aligned}$$

- In the solutions to these equations one sees the dynamics separating into an initial transient regime followed by a saturation steady state.
- There is a threshold at $\Omega = \gamma/4$ below which the solutions are monotonic functions of time and above which they exhibit oscillations.

Steady state properties

The steady state probability for the atom to be in the excited state $|2\rangle$ is

$$P_2^{ss} = \frac{1}{2}(1 + \langle\hat{\sigma}_z\rangle_{ss}) = \frac{1}{2}\frac{Y^2}{1+Y^2} \quad \text{where} \quad Y = \frac{\sqrt{2}\Omega}{\gamma}$$

- For weak driving ($Y \ll 1$) the atom settles close to its lower level, and we expect the behaviour of a classical electron oscillator.
- For *very intense illumination* the atom becomes *saturated*, with equal probability of being found in the upper and lower levels, i.e.,

$$\lim_{Y\to\infty} P_2^{ss} = \frac{1}{2}$$

Thus the atom spends 1/2 of its time in the upper state where spontaneous emission plays a significant role. *Quantum fluctuations therefore become important with intense illumination*.

Spectrum of fluorescent light

The fluorescence spectrum is defined by

$$S(\omega) = \frac{I_0(\mathbf{r})}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\langle\hat{\sigma}_+(0)\hat{\sigma}_-(\tau)\rangle_{ss}$$

where $\langle\hat{\sigma}_+(0)\hat{\sigma}_-(\tau)\rangle_{ss} \equiv \lim_{t\to\infty}\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\rangle$.
The spectrum decomposes into a *coherent component* (arising from coherent scattering), and an *incoherent component* (arising from quantum fluctuations):

$$S(\omega) = S_{coh}(\omega) + S_{inc}(\omega)$$

The coherent component is

$$\begin{aligned}S_{coh}(\omega) &= \frac{I_0(\mathbf{r})}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{i(\omega-\omega_A)\tau}\langle\tilde{\sigma}_+\rangle_{ss}\langle\tilde{\sigma}_-\rangle_{ss} \\ &= \frac{1}{2}I_0(\mathbf{r})\frac{Y^2}{(1+Y^2)^2}\,\delta(\omega - \omega_A)\end{aligned}$$

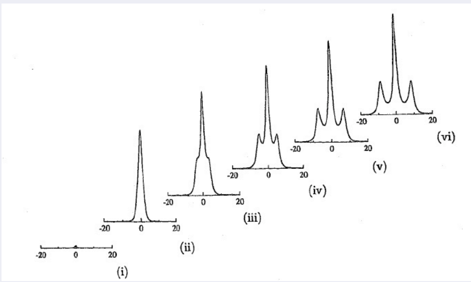
The incoherent component is

$$S_{\text{inc}}(\omega) = \frac{I_0(\mathbf{r})}{2\pi} \int_{-\infty}^{\infty} d\tau\, e^{i(\omega-\omega_A)\tau} \langle \Delta\tilde{\sigma}_+(0)\Delta\tilde{\sigma}_-(\tau)\rangle_{ss}$$

where $\Delta\tilde{\sigma}_\pm = \tilde{\sigma}_\pm - \langle\tilde{\sigma}_\pm\rangle_{ss}$.

To compute the incoherent spectrum we use the optical Bloch equations and the quantum regression formula.

The incoherent spectrum is a sum of three Lorentzian components.



- In the strong-field limit, $Y^2 \gg 1$ (iv)-(vi), where incoherent scattering dominates, this gives the well-known *Mollow, or Stark, triplet*, with the peaks located at $\omega = \omega_A$ and $\omega = \omega_A \pm \Omega$.
- The peak at $\omega = \omega_A$ has a halfwidth of $\gamma/2$, while the peaks at $\omega = \omega_A \pm \Omega$ have a halfwidth of $3\gamma/4$.

Photon correlations

To examine photon correlations we need to evaluate the second-order correlation function $G^{(2)}_{ss}(\tau)$, given in this particular case by

$$G^{(2)}_{ss}(\tau) = I_0(\mathbf{r})^2 \langle\hat{\sigma}_+(0)\hat{\sigma}_+(\tau)\hat{\sigma}_-(\tau)\hat{\sigma}_-(0)\rangle_{ss}$$

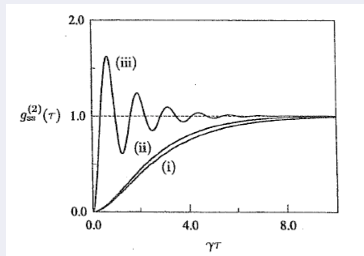
Using the quantum regression formula, we find

$$\begin{aligned} g^{(2)}_{ss}(\tau) &= \left[\lim_{\tau\to\infty} G^{(2)}_{ss}(\tau)\right]^{-1} G^{(2)}_{ss}(\tau) \\ &= 1 - e^{-(3\gamma/4)\tau}\left[\cosh(\Lambda\tau) + \frac{3\gamma/4}{\Lambda}\sinh(\Lambda\tau)\right] \end{aligned}$$

where $\Lambda = \sqrt{(\gamma/4)^2 - \Omega^2}$.

Photon antibunching in resonance fluorescence: $g^{(2)}_{ss}(0) = 0$

$g^{(2)}_{ss}(\tau)$ is plotted for increasing Y (i)-(iii):



- The fluorescent light exhibits photon antibunching due to the quantum nature of the scattering. The detection of the first photon "prepares" the atom in its ground state. Any subsequent emission must begin with an excited atom, so there is a delay corresponding to the time taken for the atom to be re-excited.

Cavity Quantum Electrodynamics

The interaction of a single two-level atom with a single mode of the electromagnetic field is the most fundamental of light-matter interactions.

In the case that the field mode is on resonance with the atomic transition we may write the Hamiltonian as $\hat{H} = \hat{H}_0 + \hat{H}_I$, with

$$\hat{H}_0 = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega\hat{\sigma}_z\,, \qquad \hat{H}_I = \hbar g\left(\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-\right)$$

This form of the interaction is known as the *Jaynes-Cummings model* (JCM).

Energy level structure

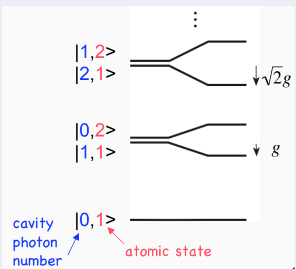
Since $[\hat{H}_0, \hat{H}_I] = 0$, the eigenstates of $\hat{H}$ can be written as linear combinations of the degenerate eigenstates of $\hat{H}_0$, $|n,2\rangle$ and $|n+1,1\rangle$, where $|n\rangle$ are number states of the field mode. In a frame rotating at frequency ω, the Schrödinger equation is

$$\tilde{H}_I \begin{pmatrix} |n,2\rangle \\ |n+1,1\rangle \end{pmatrix} = \hbar \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} |n,2\rangle \\ |n+1,1\rangle \end{pmatrix}$$

where $\Omega = g\sqrt{n+1}$.

The eigenvalues of this system are simply $\pm\hbar\Omega$, with corresponding eigenstates

$$|n,\pm\rangle = \frac{1}{\sqrt{2}}\left(|n,2\rangle \pm |n+1,1\rangle\right)$$



Dynamics: Atomic excited state probability

If the atom is initially in the excited state $|2\rangle$ and the field has exactly n photons, the probability for the atom to be in the excited state with n photons in the field at time t is

$$P_2(t) = |\langle n,2|e^{-i\tilde{H}_I t/\hbar}|n,2\rangle|^2 = \cos^2(\Omega t) = \cos^2(g\sqrt{n+1}\,t)$$

This describes the *Rabi nutation* of the atom, with Ω the *Rabi frequency*.

Quantum collapses and revivals

Consider now the case in which the field mode is initially in a *coherent state*

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle$$

If the atom is initially in the excited state $|2\rangle$, then the probability for the atom to be found in the excited state at time t is given by the Poissonian-weighted sum

$$P_2(t) = \frac{1}{2}\left[1 + \sum_n \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}\cos\left(2g\sqrt{n+1}\,t\right)\right]$$

Due to the Poisson distribution of the photon number, there is a spread in the Rabi frequencies ($\Delta n \sim \langle n\rangle^{1/2} = |\alpha|$). Consequently, the *Rabi nutation will collapse* after a certain number of oscillations due to *destructive interference* between the various cosine functions.

An approximate result valid for times $t < |\alpha|/g$ is

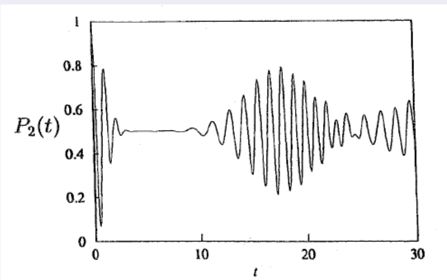
$$P_2(t) \simeq \frac{1}{2}\left\{1 + \cos\left(2g\sqrt{|\alpha|^2+1}\,t\right)\ \exp\left[-\frac{g^2t^2|\alpha|^2}{2(|\alpha|^2+1)}\right]\right\}$$

which shows that the Rabi oscillations occur under a Gaussian envelope. The characteristic time for the oscillation collapse is (for $|\alpha|^2 \gg 1$) $t_{\text{collapse}} \sim g^{-1}$, and the number of observed oscillations before the collapse is $\sim |\alpha|$.

A more accurate evaluation of the expression reveals a partial *revival of the initial oscillations* after a time

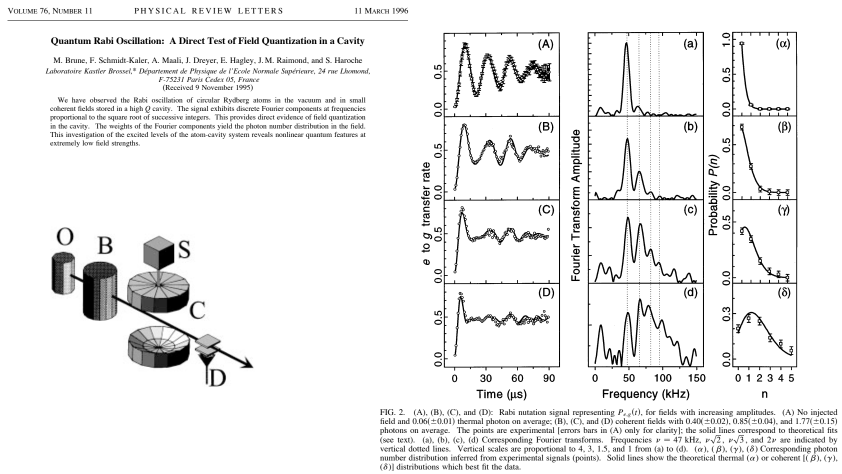
$$t_{\text{revival}} \sim \frac{2\pi}{g}|\alpha|$$

Thus a quasi-periodic burst of Rabi oscillations occurs after approximately $|\alpha|^2$ Rabi periods.



Notes

- The *existence of periodic revivals is due to the discreteness of the sum over number states*. This discrete character ensures that after some finite time the oscillating terms almost come back in phase with each other and restore the coherent oscillations.
- The rephasing is not perfect as the frequencies are irrational and thus incommensurate.
- The revivals may be considered as a *pure quantum effect* resulting from the discreteness of the harmonic oscillator spectrum.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Quantum Rabi Oscillation: A Direct Test of Field Quantization in a Cavity

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We have observed the Rabi oscillation of circular Rydberg atoms in the vacuum and in small coherent fields stored in a high Q cavity. The signal exhibits discrete Fourier components at frequencies proportional to the square root of successive integers. This provides direct evidence of field quantization in the cavity. The weights of the Fourier components yield the photon number distribution in the field. This investigation of the excited levels of the atom-cavity system reveals nonlinear quantum features at extremely low field strengths.

FIG. 2. (A), (B), (C), and (D): Rabi nutation signal representing $P_{e,g}(t)$, for fields with increasing amplitudes. (A) No injected field and 0.06(±0.01) thermal photon on average; (B), (C), and (D) coherent fields with 0.40(±0.02), 0.85(±0.04), and 1.77(±0.15) photons on average. The points are experimental [errors bars in (A) only for clarity]; the solid lines correspond to theoretical fits (see text). (a), (b), (c), (d) Corresponding Fourier transforms. Frequencies $\nu = 47$ kHz, $\nu\sqrt{2}$, $\nu\sqrt{3}$, and 2ν are indicated by vertical dotted lines. Vertical scales are proportional to 4, 3, 1.5, and 1 from (a) to (d). (α), (β), (γ), (δ) Corresponding photon number distribution inferred from experimental signals (points). Solid lines show the theoretical thermal (α) or coherent [(β), (γ), (δ)] distributions which best fit the data.

Dissipative cavity QED

To include cavity loss and atomic spontaneous emission we model the atom-cavity system with the master equation

$$\begin{aligned}\dot{\hat{\rho}} &= -i\frac{1}{2}\omega_A[\hat{\sigma}_z,\hat{\rho}] - i\omega_C[\hat{a}^\dagger\hat{a},\hat{\rho}] - ig[\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-,\hat{\rho}] \\ &\quad + \frac{1}{2}\gamma\left(2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_- - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho}\right) \\ &\quad + \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\right)\end{aligned}$$

Assuming $\omega_A = \omega_C$, the equations of motion for the mean atomic polarisation and cavity mode amplitude are (in a frame rotating at frequency ω_C)

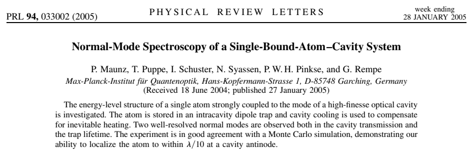
$$\begin{aligned}\langle\dot{\tilde{\sigma}}_-\rangle &= -\gamma/2\,\langle\tilde{\sigma}_-\rangle + ig\langle\tilde{\sigma}_z\tilde{a}\rangle \\ \langle\dot{\tilde{a}}\rangle &= -\kappa\,\langle\tilde{a}\rangle - ig\langle\tilde{\sigma}_-\rangle\end{aligned}$$

If the system is only *weakly excited* (e.g., by a weak probe laser driving the cavity mode), then the atom remains close to the ground state and we may set $\langle\tilde{\sigma}_z\tilde{a}\rangle \simeq \langle\tilde{\sigma}_z\rangle\langle\tilde{a}\rangle \simeq -\langle\tilde{a}\rangle$. The equations of motion for $\langle\tilde{\sigma}_-\rangle$ and $\langle\tilde{a}\rangle$ then describe *coupled oscillators*.

Normal modes

If the atom-field coupling strength is much larger than the dissipative rates, i.e., $g \gg \kappa,\gamma$, then the normal modes of the coupled atomic and cavity oscillators have frequencies $\omega_C \pm g$ (corresponding to the first two excited states of the JCM) and decay at a rate $(1/2)(\kappa + \gamma/2)$.

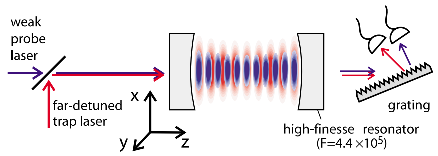
- Under these conditions, the transmission spectrum of a weak probe laser through the cavity shows resonances of width $\kappa + \gamma/2$ (FWHM) at the frequencies $\omega_C \pm g$.
- This is known as the *vacuum Rabi splitting*.

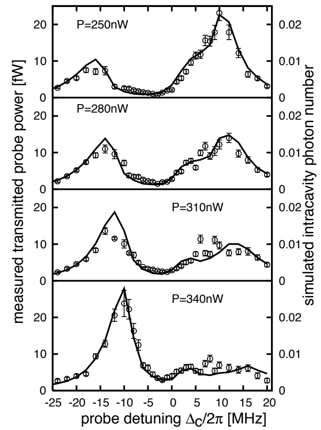

PRL 94, 033002 (2005) PHYSICAL REVIEW LETTERS week ending 28 JANUARY 2005

Normal-Mode Spectroscopy of a Single-Bound-Atom–Cavity System

P. Maunz, T. Puppe, I. Schuster, N. Syassen, P. W. H. Pinkse, and G. Rempe
Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Strasse 1, D-85748 Garching, Germany
(Received 18 June 2004; published 27 January 2005)

The energy-level structure of a single atom strongly coupled to the mode of a high-finesse optical cavity is investigated. The atom is stored in an intracavity dipole trap and cavity cooling is used to compensate for inevitable heating. Two well-resolved normal modes are observed both in the cavity transmission and the trap lifetime. The experiment is in good agreement with a Monte Carlo simulation, demonstrating our ability to localize the atom to within $\lambda/10$ at a cavity antinode.





"Bad cavity limit": cavity-enhanced spontaneous emission

The so-called "bad cavity limit" corresponds to the situation where $\kappa \gg g,\gamma$. In this case, the cavity amplitude evolves much more rapidly than the atomic polarisation, such that we may set $\langle\dot{\tilde{a}}\rangle \simeq 0$ and write

$$\langle\tilde{a}\rangle \simeq -ig\langle\tilde{\sigma}_-\rangle/\kappa$$

Assuming weak excitation of the system and substituting this expression into the equation for $\langle\dot{\tilde{\sigma}}_-\rangle$ gives

$$\langle\dot{\tilde{\sigma}}_-\rangle \simeq -\left(\gamma/2 + \frac{g^2}{\kappa}\right)\langle\tilde{\sigma}_-\rangle \equiv -\frac{\gamma}{2}(1+2C)\langle\tilde{\sigma}_-\rangle$$

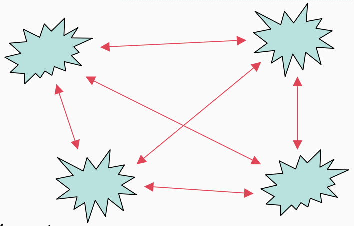
where $C = g^2/\kappa\gamma$ is the *spontaneous emission enhancement factor*.

Cavity QED: Quantum Control with Single Atoms and Single Photons

Scott Parkins
2 October 2008

Quantum Networks

Quantum node: generation, processing, & storage of quantum information (states)



Quantum channel: transfer & distribution of quantum entanglement

Matter, e.g., atoms (quantum information stored in internal, electronic states)

Matter-light interface

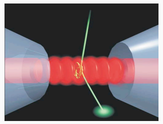
Light, e.g., single photons (quantum information stored in photon number or polarisation states)

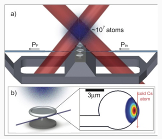
Require deterministic, reversible quantum state transfer between material system and light field

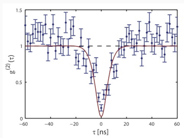
H.J. Kimble, "The quantum internet," Nature **453**, 1023 (2008)

Outline

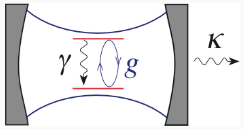
- Quantum networks
- Cavity QED
  - Strong coupling cavity QED
  - Network operations enabled by cavity QED
- Microtoroidal resonators and cold atoms
  - Cavity QED with microtoroids
  - Observation of strong coupling
  - The "bad cavity" regime
  - A photon turnstile dynamically regulated by one atom
  - Future possibilities







Cavity Quantum Electrodynamics (Cavity QED)



Atom-cavity interaction Hamiltonian

$$H = \omega_{cav} a^+ a + \omega_{atom} \sigma^+ \sigma^- + g\left(a^+ \sigma^- + \sigma^+ a\right)$$

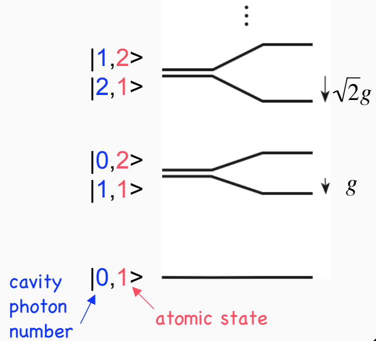
σ^+ σ^- |2> |1> 2-level atom

$g \sim \mu_{01} E$

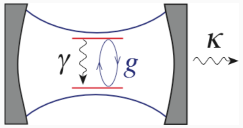
μ_{01} – atomic transition dipole moment

E – electric field per photon

$$E = \sqrt{\hbar\omega_{cav} / 2\varepsilon_0 V_{mode}}$$



Strong Coupling Cavity QED



γ - atomic spontaneous emission rate
κ- cavity field decay rate

<u>Strong dipole transition</u> in optical cavity of <u>small mode volume</u>, <u>high finesse</u>

$$g >> \kappa, \gamma$$

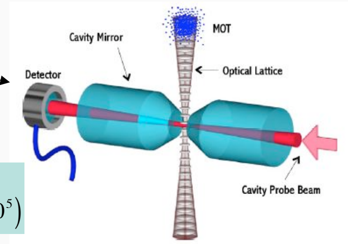
Coherent dynamics dominant over dissipative processes

- Nonlinear optics with single photons
- Strong single-atom effects on cavity response
- <u>Controllable manipulation of quantum states</u>

Experimental Cavity QED With Cold Atoms

Cavity QED with cold neutral atoms (Fabry-Perot resonators)

- H.J. Kimble (Caltech)
- G. Rempe (MPQ, Garching)
- M. Chapman (Georgia Tech)
- D. Stamper-Kurn (Berkeley)
- D. Meschede (Bonn)
- L. Orozco (Maryland)
- ...



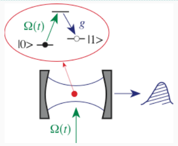
Typically $\begin{cases} g/2\pi \sim \text{few} \times 10\ \text{MHz} \\ \kappa/2\pi \sim \text{few MHz} \ \ (Q \sim 10^5) \end{cases}$

Cavity QED with trapped ions

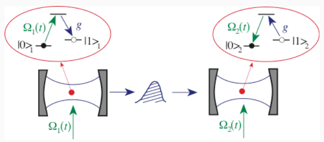
- R. Blatt (Innsbruck)
- W. Lange (Sussex)
- C. Monroe (Maryland)
- M. Chapman (Georgia Tech)
- ...

Network Operations Enabled by Cavity QED

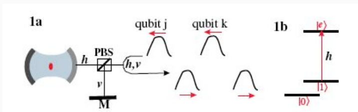
(i) Quantum State Transfer: Atom ↔ Field
- T. Wilk et al., Science **317**, 488 (2007) (expt)
- A.D. Boozer et al., Phys. Rev. Lett. **98**, 193601 (2007) (expt)



(ii) Quantum State Transfer: Node ↔ Node
- J.I. Cirac et al., Phys. Rev. Lett. **78**, 3221 (1997) (theory)

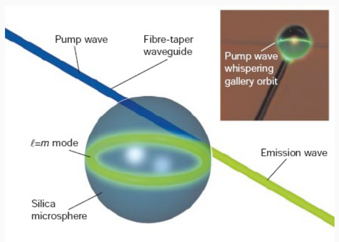


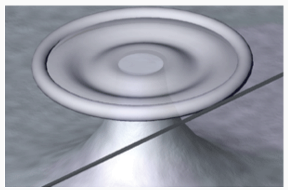
(iii) Conditional Quantum Dynamics
- L.-M. Duan & H.J. Kimble, Phys. Rev. Lett. **92**, 127902 (2004) (theory)



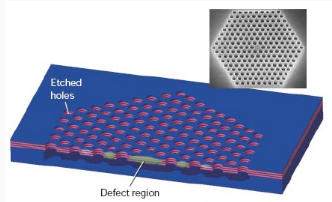
New Architectures: Optical Microcavities

K.J. Vahala, "Optical microcavities," Nature **424**, 839 (2003)



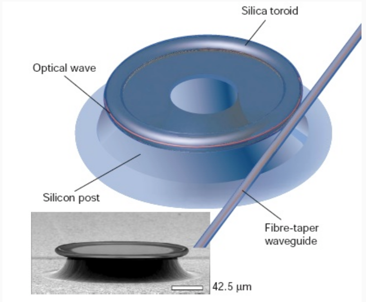


- Lithographically fabricated
- Integrable with atom chips, scalable networks



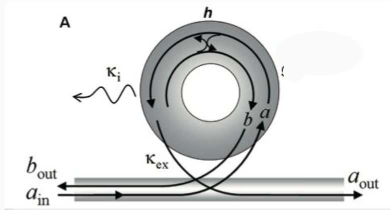
Microtoroidal Resonators + Fiber Tapers

S.M. Spillane, T.J. Kippenberg, O.J. Painter, & K.J. Vahala, "Ideality in a fiber-taper-coupled microresonator system for application to cavity quantum electrodynamics," Phys. Rev. Lett. **91**, 043902 (2003)



- Coupling through evanescent fields
- 99.97% fiber-taper to microtoroid coupling efficiency!
- Readily integrated into quantum networks
- Ultrahigh Q-factors and small mode volumes

Microtoroidal Resonator – Critical Coupling



Output fields

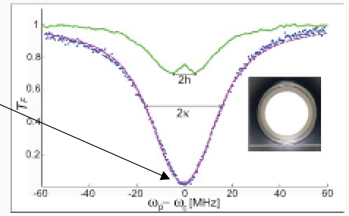
$$a_{out} = -a_{in} + \sqrt{2\kappa_{ex}}\, a$$
$$b_{out} = -b_{in} + \sqrt{2\kappa_{ex}}\, b$$

$$T_F = \frac{\langle a_{out}^+ a_{out} \rangle}{\langle a_{in}^+ a_{in} \rangle}$$

Critical coupling condition

$$\kappa_{ex} = \kappa_{ex}^{cr} = \sqrt{\kappa_i^2 + h^2}$$
$$\Rightarrow \quad T_F(\Delta_C = 0) = 0$$

(destructive interference in forward direction)



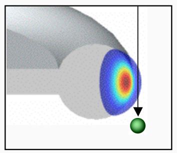
Projected Cavity QED Parameters

S.M. Spillane, T.J. Kippenberg, K.J. Vahala, W. Goh, E. Wilcut, & H.J. Kimble, "Ultrahigh-Q toroidal microresonators for cavity quantum electrodynamics," Phys. Rev. A **71**, 013817 (2005)

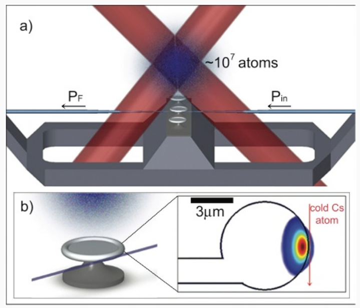
Microtoroid of major diameter 10-20 microns:

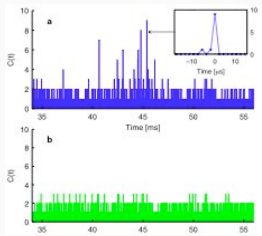
$$\begin{cases} g/2\pi \sim \text{few} \times 100\ \text{MHz} \\ \kappa_i/2\pi < 1\ \text{MHz} \quad (Q \sim 10^{8-9}) \end{cases}$$

← near surface of toroid



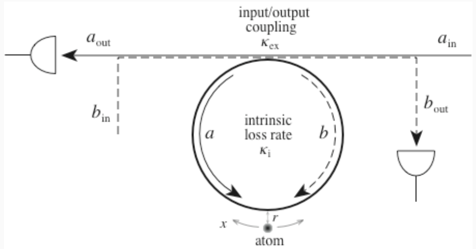
Microtoroidal Resonators + Cold Atoms





- Atoms couple to evanescent field of whispering gallery modes, "disrupt" critical coupling condition

Microtoroid Cavity QED – Basic Parameters



- Mode-mode coupling h
- Atom-field coupling

$$g_{tw}(r,x) = g_0^{tw}(r)e^{ikx}$$
$$g_0^{tw}(r) \sim e^{-kr}$$

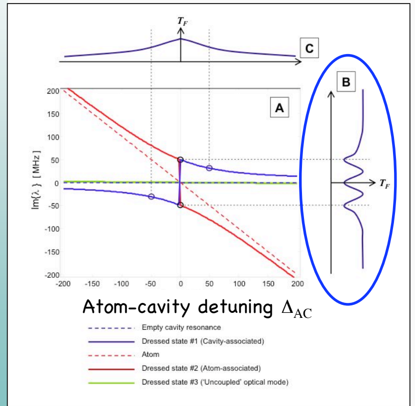
$$H = \Delta_A \sigma^+\sigma^- + \Delta_C(a^+a + b^+b) + h(a^+b + b^+a) + (E_p^* a + E_p a^+) + (g_{tw}^* a^+\sigma^- + g_{tw}\sigma^+ a) + (g_{tw} b^+\sigma^- + g_{tw}^*\sigma^+ b)$$

Probe field driving, frequency ω_p

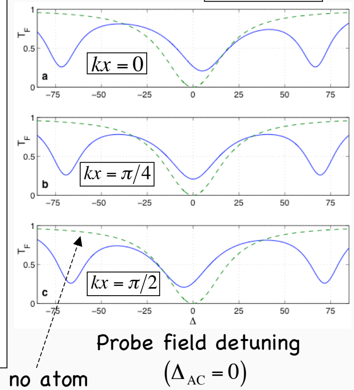
$$(\Delta_A = \omega_A - \omega_p, \quad \Delta_C = \omega_C - \omega_p)$$

Microtoroid Cavity QED

Level structure (vacuum Rabi splitting)



Forward transmission $T_F = \dfrac{\langle a_{out}^+ a_{out}\rangle}{\langle a_{in}^+ a_{in}\rangle}$



Probe field detuning $(\Delta_{AC} = 0)$

no atom

Normal Mode Picture

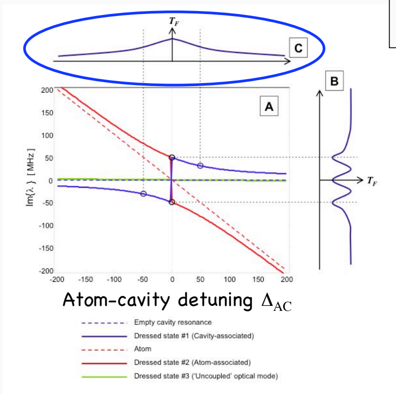
Define normal mode operators: $A = \frac{1}{\sqrt{2}}(a+b), \quad B = \frac{1}{\sqrt{2}}(a-b)$

$$H = \Delta_A \sigma^+\sigma^- + (\Delta_C + h)A^+A + (\Delta_C - h)B^+B + \frac{1}{\sqrt{2}}\left[E_p^*(A+B) + E_p(A^+ + B^+)\right] + g_A(A^+\sigma^- + \sigma^+ A) - ig_B(B^+\sigma^- - \sigma^+ B)$$

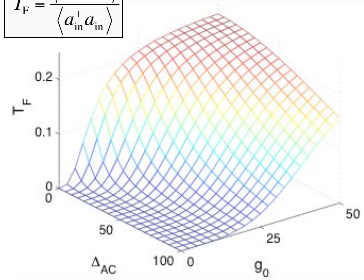
$g_A = g_0 \cos(kx)$ $\quad g_0 = \sqrt{2} g_0^{tw}$ $\quad g_B = g_0 \sin(kx)$

Normal modes ↔ standing waves around circumference of toroid

Microtoroid Cavity QED

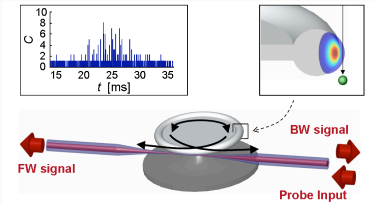


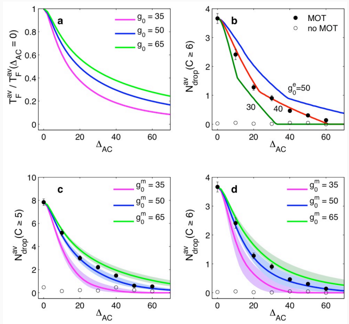
$T_F = \dfrac{\langle a_{out}^+ a_{out}\rangle}{\langle a_{in}^+ a_{in}\rangle}$



Can use dependence of T_F on Δ_{AC} to determine g_0

Observation of Strong Coupling





$$g_0^{max} \approx 2\pi\cdot 50\ \text{MHz} > \begin{cases} \kappa_{tot} \approx 2\pi\cdot 18\ \text{MHz} \\ \gamma_\perp = 2\pi\cdot 2.6\ \text{MHz} \end{cases}$$

T. Aoki, B. Dayan, E. Wilcut, W.P. Bowen, SP, T.J. Kippenberg, K.J. Vahala & H.J. Kimble, Nature **443**, 671 (2006)

"Bad Cavity" Regime

$$\kappa_{tot} \approx 2\pi\cdot 165\ \text{MHz} >> \begin{cases} g_0^{max} \approx 2\pi\cdot 70\ \text{MHz} \\ \gamma_\perp = 2\pi\cdot 2.6\ \text{MHz} \end{cases}$$ (Caltech '07)

- Theory: Adiabatic elimination of cavity modes
- Effective master equation for atomic density matrix:

$$\dot\rho_A = -i[H_A,\rho_A] + \frac{\Gamma}{2}\left(2\sigma^-\rho_A\sigma^+ - \sigma^+\sigma^-\rho_A - \rho_A\sigma^+\sigma^-\right)$$
$$H_A = \Delta'_A\sigma^+\sigma^- + \left(\Omega_0\sigma^+ + \Omega_0^*\sigma^-\right)$$

- Cavity-enhanced atomic spontaneous emission rate

$$\Gamma \sim \gamma + \frac{2g_0^2}{\kappa_{tot}} = \gamma(1+2C), \quad C = \frac{g_0^2}{\kappa_{tot}\gamma}$$

single-atom "cooperativity" parameter

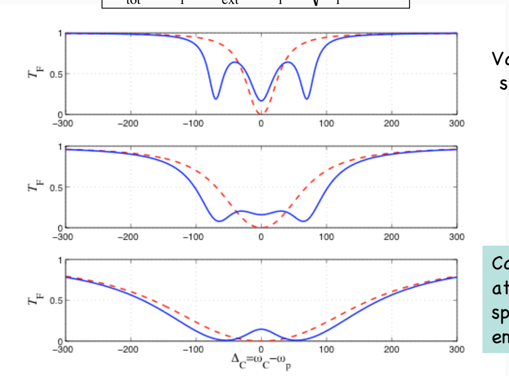
Effect of Increasing Cavity Loss

$$\kappa_{tot} = \kappa_i + \kappa_{ext}^{cr} = \kappa_i + \sqrt{\kappa_i^2 + h^2}$$

$\kappa_{tot} < g_0$

$\kappa_{tot} \approx g_0$

$\kappa_{tot} >> g_0$



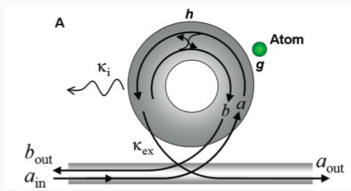
Vacuum Rabi splitting

Cavity-enhanced atomic spontaneous emission

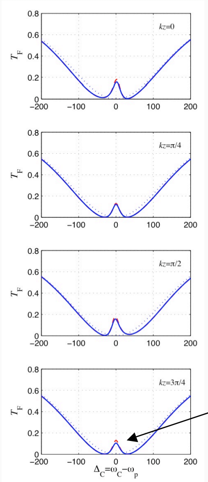
Output Fields: Bad Cavity Regime

$$a_{out} = -a_{in} + \sqrt{2\kappa_{ex}}\,a \quad\rightarrow\quad \alpha_0 + \alpha\_\sigma\_$$
$$b_{out} = -b_{in} + \sqrt{2\kappa_{ex}}\,b \quad\rightarrow\quad \beta_0 + \beta\_\sigma\_$$

$\begin{Bmatrix}\alpha_0 \\ \beta_0\end{Bmatrix}$ = coherent amplitudes <u>without</u> atom



Forward Spectra



Different azimuthal positions x

$$\begin{cases} g_0^{tw}/2\pi = 50 \text{ MHz} \\ (\kappa_i, \kappa_{ext})/2\pi = (75, 90) \text{ MHz} \\ h/2\pi = 50 \text{ MHz} \end{cases}$$

Central atomic resonance, width ≈ Γ

Note: Other photon turnstile devices

e.g.,

- J. Kim, O. Benson, H. Kan, & Y. Yamamoto, "A single-photon turnstile device," Nature **397**, 500 (1999) (semiconductor)
- K.M. Birnbaum, A. Boca, R. Miller, A.D. Boozer, T.E. Northup, & H.J. Kimble, "Photon blockade in an optical cavity with one trapped atom," Nature **436**, 87 (2005)

Blockade a structural effect due to anharmonicity of energy spectrum for multiple excitations

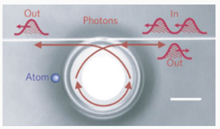
Microtoroid-atom system: blockade regulated dynamically by conditional state of one atom
→ efficient mechanism, insensitive to many experimental imperfections

A Photon "Turnstile"

Bad cavity regime

$$a_{out} \rightarrow \alpha_0 + \alpha_- \sigma^-$$
$$b_{out} \rightarrow \beta_0 + \beta_- \sigma^-$$

- Critical coupling: $\alpha_0(\Delta_C \approx 0) \approx 0,\ \beta_0(\Delta_C \approx 0) \neq 0$
- \`1st' photon transmitted into a_{out} can only originate from atom
- Emission projects atom into ground state
- \`2nd' photon cannot be transmitted until atomic state regresses to steady-state, time scale $1/\Gamma$



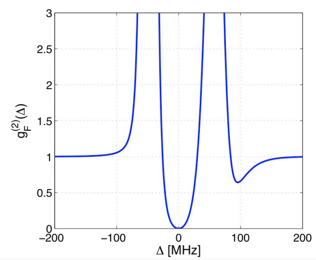
⇒ excess photons \`rerouted' to b_{out}

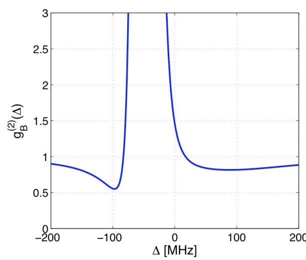
Microtoroid-atom system only transmits photons in the forward direction one-at-a-time

Signatures: Intensity Correlation Functions

$$g_F^{(2)} = \frac{\langle (a_{out}^+)^2 a_{out}^2 \rangle}{\langle a_{out}^+ a_{out} \rangle^2}, \quad g_B^{(2)} = \frac{\langle (b_{out}^+)^2 b_{out}^2 \rangle}{\langle b_{out}^+ b_{out} \rangle^2}$$

(probabilities of "simultaneous" photon detections)



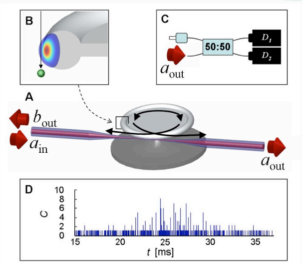


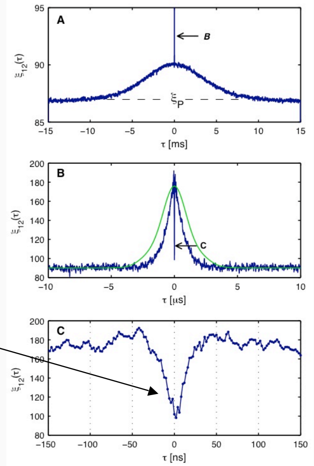
antibunching at $\Delta \approx 0$

$$\langle (a_{out}^+)^2 a_{out}^2 \rangle \sim \langle \sigma^{+2} \sigma^{-2} \rangle = 0$$

bunching at $\Delta \approx 0$

Experiment (Caltech '07)



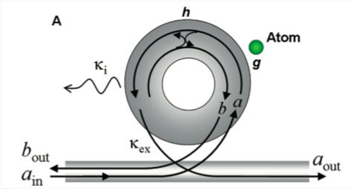


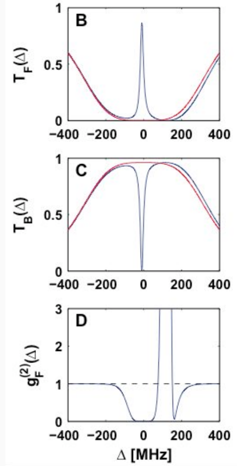
- Cross correlation $\xi_{12}(\tau)$
- $\xi_{12}(\tau) > \xi_{12}(0)$ a prima facie observation of nonclassical light

In the Future …

- Minimise intrinsic losses
  $\kappa_i << \kappa_{ex}$
- Large mode-mode coupling h

$\Rightarrow$ Near-ideal input/output

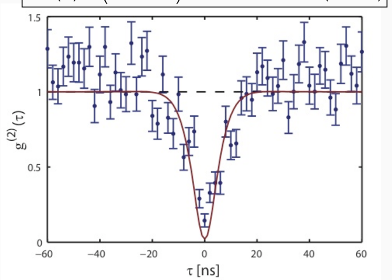




Observation of Antibunching/Turnstile Effect

- Analysis of single and joint detections at $D_{1,2}$ conditioned on single atom transit

$$g_F^{(2)}(\tau) \approx \left(1 - e^{-\Gamma\tau/2}\right)^2, \quad 1/\Gamma \approx 2.8 \text{ ns } (C \sim 5)$$



"Blockade" effect robust, e.g., requires only

$$\frac{2g(\vec{r})^2}{\kappa_{tot}\gamma} > 1$$

Dayan, Parkins, Aoki, Kimble, Ostby & Vahala, "A Photon Turnstile Dynamically Regulated by One Atom," Science **319**, 1062 (2008)

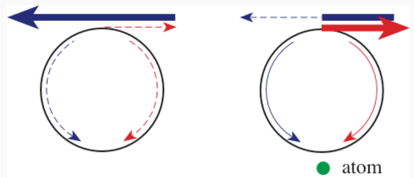
Microtoroid + Atom: Over-Coupled Regime

Bad cavity regime

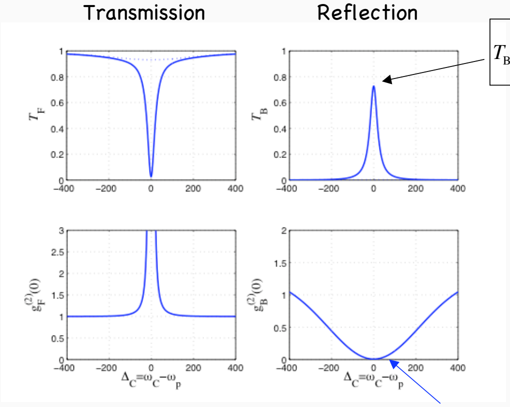
$$a_{out} \to \alpha_0 + \alpha\_\sigma^-$$
$$b_{out} \to \beta_0 + \beta\_\sigma^-$$

- Strong over-coupling: $\kappa_{ex} >> h,\ \kappa_i \quad (\kappa_{tot} \approx \kappa_{ex})$
- No atom ($\alpha\_ = \beta\_ = 0$): strong transmission, small reflection ($\beta_0 \approx 0$)
- With atom: destructive interference between α_0 and α_σ^-

  $\Rightarrow$ strong reflection, small transmission



Spectra and Correlations: Over-Coupled Regime



$$T_B(\Delta_C=0)\approx\left(\frac{\kappa_{ex}}{\kappa_{tot}}\right)^2\left(\frac{2C}{1+2C}\right)^2$$

$\kappa_{tot}\approx\kappa_{ex}$

$$C\sim\frac{g_0^2}{\kappa_{tot}\gamma}>>1$$

Single atom cooperativity

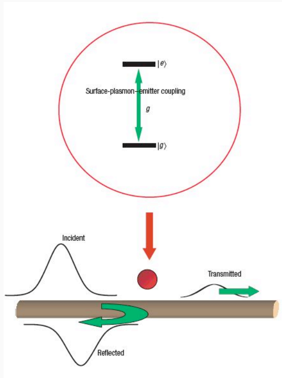
antibunching in reflected field

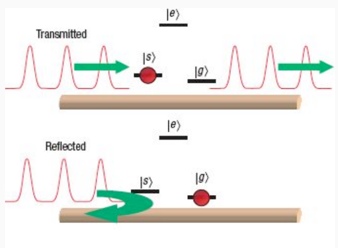
... and beyond

- Controlled interactions of single-photon pulses
- Trapping of atoms close to toroid
- Multiple toroid+atom systems
  - → Spin networks
  - → Scalable quantum information processing on atom chips

Single Photon "Transistor"

D.E.Chang, A.S. Sorensen, E.A. Demler, & M.D. Lukin, "A single-photon transistor using nanoscale surface plasmons," Nature Physics **3**, 807 (2007)





Microdisk-Quantum Dot Systems

K. Srinivasan & O. Painter, "Linear and nonlinear optical spectroscopy of a strongly coupled microdisk-quantum dot system," Nature **450**, 862 (2007)