Conformal Quantum Field Theory

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Abstract. Conformal symmetry allows to derive a priori properties of relativistic quantum field theories, without the need to specify a model Lagrangian. In four spacetime dimensions, these properties basically concern the general form of correlation functions, and positivity conditions allow to constrain the free parameters. They go far beyond this in two dimensions, allowing for complete solutions or (discrete) classifications. The lecture introduces several facets of the amazing structures arising in 2D CFT and their surprising relations to various areas of mathematics.

0 Disclaimer

Most of you are presumably familiar with quantum field theory in terms of path-integrals and Feynman diagrams, that are well-suited tools to compute scattering cross sections in particle physics. You will not see anything of these in my lectures. In contrast, I want to acquaint you with a more intrinsic approach that rather focusses on algebraic properties of quantum fields, independent of specific model input. These algebraic properties determine the physics. They include commutation relations, that may turn out very different from the "canonical commutation relations" imposed for free fields.

Conformal symmetry, especially in two spacetime dimensions, considerably constrains the possible form of these algebraic structures, so that many nontrivial solutions can be found explicitly. These may give us only a faint impression of the vast range of possibilities to be expected in the general case without special symmetries.

1 QFT

Quantum Field Theory is "the" paradigm for our understanding of the fundamental constituents of matter, in particular Particle Physics [1].

It appears in many different versions, that all share the following features:

- Relativistic invariance
- Locality and Causality
- Particles and scattering theory

In spite of its tremendous successes, QFT is not a "final theory". Its problems include:

- UV divergences in perturbation theory ⇒ Renormalization
- Indefinite-metric "Hilbert" spaces ⇒ BRST (or other)
- Various kinds of infrared problems
- Divergence of the perturbative expansion ⇒ ???
- Difficult control of non-perturbative effects
- Quark confinement and hadronization
- Inclusion of "gravitons" ⇒ ???
- A future theory of Quantum Gravity must fundamenally transcend QFT, because "spacetime", which is the fixed stage of QFT, becomes itself the dynamical player of QG.

Whether a QFT describing the Standard Model exists as a mathematically complete and consistent theory, is not known.

Most attempts at a construction rely on the "quantization of a classical model", defined by a Lagrangian. These provide computational recipes for scattering cross sections, that are perfectly confirmed by experiment. But they are known to be only approximations of non-convergent perturbative expansions!

"Axiomatic approaches" have been pursuited for more than 50 years, with the aim to "explore the space of theories" without specific model assumptions. The idea is that "the true theory" should fit into such a scheme, no matter how the actual computations are done. An axiomatic scheme should allow structural apriori insights, that admit predictions without detailed computation. An example is the PCT theorem, that was first known to hold "by inspection of all possible Lagrangians"; but in the axiomatic setting it became clear to be a consequence of more fundamental properties without the assumption of an underlying Lagrangian.

In these lectures, I tacitly adopt the most widely accepted axiomatic setting formulated by Wightman [2] ("Hilbert space, locality, covariance, vacuum") without exhibiting the technicalities.

2 Conformal QFT

2.1 Motivation

Conformal symmetry has a long history in geometry, starting from the observation that the map

$$\vec{x} \mapsto \frac{\vec{x}}{|\vec{x}|^2}$$

preserves angles and maps planes to spheres. The first property was used in cartography ("Mercator projection"), and the second property entered classical physics, e.g., in the "method of images" to compute the electric field of charges near conducting spheres. An extensive review of this classical background can be found in [3]. Conformal QFT (= CFT) has many different motivations [4].

First, classical Maxwell theory is conformally invariant, so why not quantum Maxwell theory? Of course, massive charged particles spoil conformal invariance, but possibly QED becomes conformally invariant in the high-energy limit where the electron mass should be negligible? This expectation was disproved by the discovery of the "running coupling constant" by which dimensionless parameters do not scale trivially. Asymptotically free theories become free in the high-energy limit: the limiting theory is conformally invariant but tells us nothing about the actual theory. This, historically earliest, motivation has faded away.

Second, the AdS-CFT correspondence has renewed interest in CFT in four (or more) spacetime dimensions because it is believed to be "dual" to some gravity theory in one dimension more.

Third, lattice (spin) systems at critical points (UV fixed points) may become conformally invariant Euclidean field theories, which in turn may be "Wick rotated" Lorentzian CFTs. Indeed, some of the most studied models in two dimensions, including the Ising model, belong to this class.

Fourth, String Theory can be formulated in terms of an internal "worldsheet dynamics" which is for itself a two-dimensional CFT. Classifications of the latter have a bearing on the possible degrees of freedom (geometric or non-geometric) of the associated ST.

Fifth, CFT is just "relativistic QFT with extra symmetries", because the conformal group contains the Poincaré group. Every general result that can be proven in QFT, is also true in CFT. CFT can therefore be used as a testing ground for general QFT, where the additional symmetry allows stronger apriori structure results and sharper classifications. It is an ideal stage to

"explore the space of theories".

Indeed, in two spacetime dimensions, these results are so strong that nontrivial models can be rigorously constructed without relying on perturbation theory, and not even invoking an underlying Lagrangian. This last one is my personal main motivation to study conformal QFT [5].

An interesting side aspect is the fact that "particles" are not a good concept in CFT. "Particles" in an interacting QFT are only asymptotic features of states, which can be isolated because massive excitations of different momentum travel at different velocities and will separate at asymptotic times. In contrast, massless particles like photons do not separate, and cannot always be identified as individual objects. This well-known "IR problem" in QED leads to the necessity of considering inclusive cross sections involving arbitrary numbers of soft photons of arbitrarily low energy.

Therefore, in conformal QFT, the quantum fields stand in the foreground, rather than being only tools to describe the scattering of particles. One is interested in their (algebraic) properties, whose understanding brings QFT back to its roots.

A standard text book is [4].

2.2 The conformal group: geometry and group theory

A standard text book is [4]. The conformal group $\mathrm{Conf}_{1,3}$ (in four spacetime dimensions) extends the Poincaré group by the dilations $x \mapsto \lambda x$ and the commutative group of "special conformal transformations"

$$x \mapsto C_b(x) = \frac{x - x^2\, b}{1 - 2(bx) + b^2 x^2} \quad (b \in \mathbb{R}^4).$$

These are the transformations $x \mapsto x' = g(x)$ that preserve the metric up to a local scale factor:

$$dx'^2 = \omega_g(x)^2 \cdot dx^2.$$

In particular, conformal transformations preserve lightlike directions. The invariant property of spacetime is called "conformal structure" = metric modulo factors.

The connected group generated by these transformations is isomorphic to $SO(2,4)_0$. It contains the "conformal inversion"[1] [2]

$$I(x) = \frac{(-x^0, \vec{x})}{x^2}, \quad I \circ I = \mathrm{id},$$

and the special transformations arise by conjugation with I from the translations:

$$C_b(x) = I(I(x) + \widetilde{b}), \quad \widetilde{b} = (b^0, -\vec{b}).$$

One should worry about the singularities of conformal transformations. This issue can be resolved by noticing that Minkowski spacetime is just a chart of the "Dirac manifold" on which the action of $\mathrm{Conf}_{1,3}$ is perfectly regular. The singularities are then only an artifact of the circumstance that the Minkowski chart misses a subset of measure zero of the Dirac manifold (the "points at infinity"). The Dirac manifold $\mathbb{M}^{1,3}$ is the "projective null cone in $2+4$ dimensions":

$$\big\{X = (y^{-1}, y^0, z^1, \ldots, z^4) \in \mathbb{R}^{2,4} \backslash \{0\} : X \cdot X \equiv \vec{y}^2 - \vec{z}^2 = 0\big\}/_{x \sim \lambda x\, (\lambda \neq 0)} = (S^1 \times S^3)/\mathbb{Z}_2.$$

[1]For dimensional reasons, this formula should be understood with x meaning x/R for some length scale R. Because of scale invariance, R is arbitrary, and will be suppressed in the sequel. Thus coordinates and momenta become dimensionless quantities.

[2]The metric convention is $\eta_{\mu\nu} = (+, - \cdots -)$. In the literature one often finds instead $\widetilde{I}(x) = \frac{x}{x^2}$ and $C_b(x) = \widetilde{I}(\widetilde{I}(x) - b)$. This is also correct, but $\widetilde{I}$ does not belong to the connected group because it differs from I by a time reversal, and is not a priori expected to be a symmetry of QFT.

$SO(2,4)$ acts in the obvious way, and the standard Minkowski coordinates are

$$x^\mu = \frac{(y^0, z^1, z^2, z^3)}{y^{-1} + z^4}.$$

Exercise: under conformal transformations, the tensor $R^{\mu\nu}(z) = z^2\eta^{\mu\nu} - 2z^\mu z^\nu$ transforms as

$$R(g(x) - g(y)) = (J_g(x) \otimes J_g(y))R(x - y),$$

where $(J_g)_\mu^\nu = \frac{\partial g(x)^\nu}{\partial x^\mu}$ is the Jacobian of g. Because $dg(x)^2 = \omega_g(x)^2 dx^2$, one has $J_g(x) = \omega_g(x) \cdot \Lambda_g(x)$, where $\Lambda_g(x)$ is a Lorentz matrix; hence $\det(J_g(x)) = \omega_g(x)^4$. Taking the determinant, it follows that finite Lorentz distances change by a multiplicative factor:

$$(g(x) - g(y))^2 = \omega_g(x)\omega_g(y) \cdot (x - y)^2.$$

Taking $x - y$ infinitesimal, gives back $dg(x)^2 = \omega_g(x)^2 dx^2$.

2.3 The conformal group: representation theory

The commutation relations of the (self-adjoint) generators (in a unitary representation) are

$$i[P_\mu, P_\nu] = 0, \quad i[P_\mu, M_{\kappa\lambda}] = \eta_{\mu\lambda}P_\kappa - \eta_{\mu\kappa}P_\lambda, \quad i[M_{\kappa\lambda}, M_{\mu\nu}] = \eta_{\kappa\mu}M_{\lambda\nu} \pm \ldots;$$

$$i[K_\mu, K_\nu] = 0, \quad i[M_{\kappa\lambda}, K_\mu] = \eta_{\kappa\mu}K_\lambda - \eta_{\lambda\mu}K_\kappa, \quad i[P_\mu, K_\nu] = -2\eta_{\mu\nu}D + 2M_{\mu\nu};$$

$$i[D, P_\mu] = P_\mu, \quad i[D, K_\mu] = -K_\mu, \quad i[D, M_{\kappa\lambda}] = 0.$$

In QFT, a unitary symmetry is in general a representation of the universal covering group. $SO(2,4)_0$ is not simply connected because of the "timelike" subgroup $SO(2)$. Its generator $L_0 = \frac{1}{2}(P_0 + K_0)$ is called the "conformal Hamiltonian". It is positive in a positive-energy representation, because $P_0 \geq 0$ and $K_0 = IP_0I \geq 0$. The spectrum of L_0 is discrete, because $e^{i2\pi L_0}$ (the representative of the rotation by 2π) commutes with $U(g)$ and hence is a multiple of $\mathbf{1}$ in every irreducible representation. It is more informative than the spectrum of the true Hamiltonian P_0 (which is always $\mathbb{R}_+$.) The lowest eigenvalue of L_0 in an irreducible unitary positive-energy repn is called the "scaling dimension" d (see below).

The nontrivial irreducible unitary positive-energy repns are labelled by a pair (j_1, j_2) of "spin" quantum numbers (corresponding to dotted and un-dotted spinor indices, i.e., a matrix representation of the Lorentz group) and the scaling dimension d, taking real values subject to the "unitarity bound" $d \geq j_1 + j_2 + 2$ if $j_1, j_2 \neq 0$, resp. $d \geq j_1 + j_2 + 1$ if $j_1 = 0$ or $j_2 = 0$ [16].

2.4 Conformal QFT in four spacetime dimensions

In a field theory with a conserved stress-energy tensor: $\partial_\nu T^{\mu\nu} = 0$, the four-momentum

$$P^\mu = \int_{x^0=t} T^{\mu 0}(x)\, d^s x$$

is independent of t. If (and only if) $T^{\mu\nu}$ is also symmetric, then also $J^\nu_{\kappa\lambda} = x_\kappa T^\nu_\lambda - x_\lambda T^\nu_\kappa$ is conserved, and

$$M_{\kappa\lambda} = \int_{x^0=t} J^0_{\kappa\lambda}(x)\, d^s x$$

is independent of t. If (and only if) $T^{\mu\nu}$ is also traceless, $T = T^\kappa_\kappa = 0$, then also $J^\nu = x_\kappa T^{\kappa\nu}$ conserved, and

$$D = \int_{x^0=t} J^0(x)\, d^s x$$

is independent of t. In this case, also $J^\nu_\mu = 2x_\mu x_\kappa T^{\kappa\nu} - x^2 T^\nu_\mu$ is conserved, and

$$K_\mu = \int_{x^0=t} J^0_\mu(x)\, d^s x$$

is independent of t.

In conformal QFT, the Hilbert space carries a unitary positive-energy repn of the conformal group, whose generators arise as above from a conserved traceless and symmetric quantum stress-energy tensor.[3]

Covariant quantum fields transform according to

$$U(g)\phi(x)U(g)^* = R(J_g(x))^t \phi(g(x)).$$

Here, ϕ may be a Lorentz tensor or spinor, $J_g(x) = \omega_g(x)\Lambda_g(x)$ is the Jacobi matrix, and $R(\omega\Lambda) = \omega^d \cdot R(\Lambda)$ is a matrix representation of the group of Lorentz transformations and dilations, depending on the quantum numbers of the field. In particular, for the scale transformations, $J_\lambda = \lambda \cdot \mathbf{1}_4$ is represented by $R(J_\lambda) = \lambda^d \cdot \mathbf{1}$:

$$U(g)\varphi(x)U(g)^* = \lambda^d \cdot \varphi(\lambda x).$$

The parameter d is called the scaling dimension of the field (because its n-point correlation functions are homogeneous distributions in all coordinates of total degree $-n \cdot d$, see below). The massless scalar Klein-Gordon field has $d = 1$ ($d = \frac{1}{2}(D-2)$ in D dimensions) and $R(J) = (\det J)^{\frac{1}{4}}$, the Maxwell field strength has $d = 2$ and $R(J) = J \otimes J$. The stress-energy tensor has $d = 4$ (or $d = D$ in D spacetime dimensions) and $R(J) = (\det J)^{\frac{1}{2}} \cdot J \otimes J$.

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[3] The quantization of a classical conformal field theory may exhibit anomalies, so that the quantum stress-energy tensor may fail to be traceless, and the QFT fails to be conformal. We do not consider this case here.

The infinitesimal transformation laws have the form

$$i[P_\mu \phi_A(x)] = \partial_\mu \phi_A(x), \quad i[D, \phi_A(x)] = ((x^\mu \partial_\mu) + d)\phi_A(x),$$

$$i[M_{\mu\nu}, \phi_A(x)] = (x_\mu \partial_\nu - x_\nu \partial_\mu)\phi_A(x) + (L_{\mu\nu})_A^B \phi_B(x),$$

$$i[K_\mu, \phi_A(x)] = (2x_\mu(x \cdot \partial) - x^2 \partial_\mu + 2d_A x_\mu)\phi_A(x) + 2x^\nu (L_{\mu\nu})_A^B \phi_B(x),$$

where $L_{\mu\nu}$ is the infinitesimal matrix representation associated with $R(\Lambda)$.

Notice that derivatives $\partial_\kappa \phi_A$ of conformal fields transform in the same way under P_μ, D and $M_{\mu\nu}$ (with d increased by one and L adjusted for the extra Lorentz index); but not under K_μ (hence L_0, hence I). Therefore, the conformal transformation law allows to distinguish "quasi-primary" fields (commutators as above) and their derivatives ("secondary" fields).[4]

For a quasi-primary field of dimension d, one computes that $\phi(x)\Omega =$ is an (improper) eigenvector of the operator $e^{2\pi i L_0}$ with eigenvalue $e^{2\pi i d}$. This means that the spectrum of L_0 is contained in $d + \mathbb{Z}$. Moreover, $e^{-P_0}\phi(0)\Omega$ is a ground state of L_0 with eigenvalue ("lowest weight") $= d$. These facts relate the spectrum of the conformal Hamiltonian on the full Hilbert space to the field content of the model.

In QFT, one is interested in the vacuum correlation functions ("n-point fns")

$$(\Omega, \phi_1(x_1) \dots \phi_n(x_n)\Omega),$$

because they completely determine the theory. Conformal symmetry imposes strong constraints on their most general form. If the vacuum is invariant: $U(g)\Omega = \Omega$, then one can show that every n-point function of tensor fields is a product of tensors $R^{\mu\nu}(x_i - x_j)$, (inverse)[5] powers of $x_{ij}^2 = (x_i - x_j)^2$, and an arbitrary function of the conformally invariant "cross-ratios"

$$\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}.$$

For two and three points, there are no cross-ratios, hence 2- and 3-point functions are a priori determined up to scalar factors. In particular, for vector fields,

$$(\Omega, J^\mu(x) J^\nu(y)\Omega) \sim \frac{R^{\mu\nu}(x-y)}{((x-y)^2)^{d+1}}.$$

Exercise: Show that if J^μ is conserved, then $d = 3$.

[4]The term "primary" is reserved for a more specific property in $d = 2$ dimensions, see below.

[5]Here, all inverse powers of x_{ij}^2 $(i < j)$ are understood as distributions by giving x_{ij}^0 an infinitesimal imaginary part.

Also the converse is true, and the same holds for symmetric traceless conserved tensor currents of rank r, with $d = r + 2$. Thus, the scaling dimensions of conserved currents are "protected" by conformal symmetry and cannot vary, say, as a function of a coupling constant. Clearly, the stress-energy tensor has dimension $d = 4$.

The functions of the cross-ratios are model-dependent. They are not determined by conformal symmetry; but they cannot be completely arbitrary for two reasons: first, correlation functions must be symmetric under the exchange of two adjacent field entries at spacelike distance; second, correlation functions are scalar products between Hilbert space vectors, and must be subject to positivity conditions.

Example: the 4-point fn of a scalar field of dimension $d = 2$ is of the form

$$\frac{F(u,v)}{(x_{12}^2 x_{34}^2)^2}, \quad \text{where} \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2},\ v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$

When x_{12} resp. x_{23} is spacelike, locality requires invariance under $x_1 \leftrightarrow x_2$ resp. $x_2 \leftrightarrow x_3$:

$$F(u,v) = F(u/v, 1/v) \quad \text{resp.} \quad F(u,v) = u^d \cdot F(1/u, v/u).$$

A very simple, rational solution is

$$F(u,v) = A(1 + u^2 + u^2/v^2) + B(u + u/v + u^2/v).$$

Positivity requires that

$$\int d^4x_1 \dots d^4x_4\, \overline{f(x_2,x_1)} f(x_3,x_4) \cdot \frac{F(u,v)}{(x_{12}^2 x_{34}^2)^2} \geq 0$$

for arbitrary smearing functions f. This imposes further constraints on the parameters A and B, that are notoriously difficult to evaluate. See the next section.

2.5 OPE, PWE and conformal bootstrap

The operator product expansion is an asymptotic expansion of the form

$$\phi_A(x)\phi_B(y) = \sum\nolimits_C g_{AB}^C(x-y) \cdot \phi_C(y) = \sum\nolimits_{C\,\mathrm{qp}} g_{AB}^C(x-y,\partial) \cdot \phi_C(y),$$

where the first sum extends over secondary fields, and the second sum only over quasi-primary fields. By scale invariance, the coefficient functions $g_{AB}^C(x-y)$ must be homogeneous of degree $d_C - d_A - d_B$, so the higher contributions with large d_C will be suppressed at short distances. Because $d_C \geq 0$, the strongest singularity is $O((x-y)^{-d_A-d_B})$. This explains why, in the above example, there are no terms more singular than u^{-2}.

The quasi-primary form of the OPE allows a decomposition of the 4-point function into its contributions from the irreducible representations of $\mathrm{Conf}_{1,3}$. These contributions are called "partial waves". They can be thought of as the result of inserting a projection onto the repn of $\mathrm{Conf}_{1,3}$ corresponding to the quasi-primary field ϕ_C: $(\Omega, \phi_1\phi_2 P_C \phi_3\phi_4\Omega)$, and are determined as solutions to certain eigenvalue differential equations, obtained by inserting Casimir operators of $\mathrm{Conf}_{1,3}$ in the middle. For scalar fields, they were computed in [6], and the decomposition of general rational 4-point functions was worked out in [7]. Positivity requires that all coefficients are non-negative. Thus, one can test the positivity of a given (local) 4-point function.

A more efficient method is to select a quasi-primary contribution ϕ_C within the OPE by certain differential operators in x and y, followed by putting $x = y$. These operators are also determined by conformal symmetry, and were worked out in [8].

Applying such operators to the first and the last pair of fields in a correlation function $(\Omega, \phi_A\phi_B\phi_B\phi_A\Omega)$, one obtains a 2-point function of the selected quasi-primary. Its coefficient must be positive. The most trivial such operator (that selects the quasi-primary field $\mathbf{1}$) is just multiplication with $((x-y)^2)^2$ and putting $x = y$. Its application to the above example gives simply $F(u,v)|_{u=0,v=1} = A$, hence A must be positive. The operator $(\partial_x\partial_y) \circ ((x-y)^2)^2 \ldots |_{x=y}$ selects the quasi-primary scalar field of dimension 2. Its application gives

$$\frac{\partial_u F(u,v)|_{u=0,v=1}}{((x_{14})^2)^2} = \frac{2B}{((x_{14})^2)^2},$$

hence $B \geq 0$. Proceeding, one also gets upper bounds for B relative to A.

This method can be recursively applied to $2n$-point functions, and also triangle inequalities between scalar products can be tested, resp. turned into conditions on the undetermined parameters.

The "new conformal bootstrap" [9] since 2008 is a program in a very similar spirit. But here, positivity is automatic and locality is tested via the so-called "crossing symmetry": The two symmetries of $F(u,v)$ above imply (for scalar fields of dimension d, and x_1, x_2, x_3 all spacelike separated) a third one:

$$v^d F(u,v) = u^d F(v,u).$$

By performing the OPE both for $\phi(x_2)\phi(x_1)\Omega$ and for $\phi(x_3)\phi(x_4)\Omega$, and collecting the contributions of a quasi-primary field ϕ_C, one finds the general structure[6]

$$F(u,v) = 1 + \sum\nolimits_C \lambda_C^2 \cdot F_C(u,v).$$

[6]The "conformal blocks" F_C are the same as the previous partial waves without the denominator $(x_{12}^2 x_{34}^2)^d$. Hence they are also fixed by conformal symmetry, and known in the scalar case. The term 1 is the contribution from the trivial field $\mathbf{1}$ (assuming a proper normalization of ϕ). The coefficients are positive because $(\Omega, \phi_C^* \phi_C \Omega)$ is positive.

This representation of F turns the crossing symmetry into the "sum rule" condition

$$\sum_C \lambda_C^2 \cdot \frac{v^d F_C(u,v) - u^d F_C(v,u)}{u^d - v^d} \stackrel{!}{=} 1.$$

While the conformal blocks are known, both the list of quasi-primary fields ϕ_C appearing in the sum, and their coefficients λ_C^2 are model-dependent. The idea is to determine the possible "field contents" of a conformal QFT by solving the sum rule with positive coefficients $\lambda_C^2 \geq 0$ – which turns out to very restrictive.

This program can be efficiently automatized, and has been particularly successful in three spacetime dimensions. Specifying the scaling dimensions of a few lowest nontrivial fields, numerical studies show that – if the sum rule can be solved at all with the given input – the dimensions of all other fields and their coefficients are uniquely fixed. This substantially narrows down the "space of theories" to a few isolated islands (presumably points) [10] and a continuum far away. One can indeed identify some well-known models originally defined by different methods, like critical lattice models; but it should be emphasized that the conformal bootstrap is entirely intrinsic and does not refer to any Lagrangian or (lattice) Hamiltonian.

2.6 AdS-CFT

Four-dimensional Anti-deSitter spacetime $\mathrm{AdS}_{1,3}$ is a cosmological solution of Einstein's vacuum equations with negative cosmological constant Λ and negative spatial curvature. With $R^2 = 3/|\Lambda|$, the metric can be written as

$$ds^2 = (r^2/R^2 + 1)dt^2 - \frac{dr^2}{r^2/R^2 + 1} - r^2 d\Omega^2.$$

It can be represented as the hyperboloid $X \cdot X = -R^2$ in $\mathbb{R}^{2,3}$. The same is true with five-dimensional $\mathrm{AdS}_{1,4}$, embedded in $\mathbb{R}^{2,4}$. In particular, its connected isometry group is $SO(2,4)_0$ – the same as the conformal group $\mathrm{Conf}_{1,3}$ in four dimensions.

A QFT on an $\mathrm{AdS}_{1,4}$ background comes with a unitary representation of $SO(2,4)_0$ with an invariant vacuum vector. Moreover, because the conformal Hamiltonian as a generator of $SO(2,4)_0$ is the generator of the true AdS time evolution, a positive conformal Hamiltonian is equivalent to a positive AdS Hamiltonian.

Thus, QFT on five-dimensional AdS and CFT on four-dimensional Minkowski come with the same symmetry group and the same class of unitary representations. Is there a relation among the two? Obviously, the fields cannot be directly identified because of the different spacetime dimension, and a different transformation law

$$U(g)\Phi(X)U(g)^* = D(g)^t \Phi(gX)$$

on AdS as compared to the conformal transformation law on Minkowski spacetime.

Maldacena [11] has in 1998 formulated a stunning and seminal conjecture: a string theory in an asymptotically AdS background should be "dual" to a conformal field theory. Here, Minkowski spacetime is identified with the "boundary" of AdS at spacelike infinity. Writing the metric as

$$ds^2 = \frac{R^2}{z^2} \cdot \big[dx_\mu dx^\mu - dz^2\big] \qquad (z > 0),$$

the boundary is given by $z = 0$ (think of a distance $r = \frac{1}{z} \to \infty$) where the prefactor diverges and only the conformal structure of the Minkowski metric $dx_\mu dx^\mu$ survives. More precisely, this boundary is the Dirac manifold (Sect. 2.2). The "duality" is formulated by a relation between the path-integral prescriptions: the sources of the CFT are identified with prescribed boundary values of the AdS field (see below).

Witten [12] has supplemented this idea by a simple model with free fields in a fixed AdS background. It shows on the one hand the AdS-CFT relation is not necessarily a feature of string theory; and on the other hand that a QFT on AdS with a (free) Lagrangean may be dual to a CFT that does not have a Lagrangean (known as "generalized free field").

Bertola et al. [13] have extended this model to a general result: Let $\Phi(x, z)$ be an AdS-covariant quantum field on AdS, such that $z \to 0$ is the boundary limit and x a Minkowski coordinate. Then, with a suitable scaling function $N(z)$, the limit

$$\varphi(x) = \lim_{z\to 0} N(z)\Phi(x, z)$$

is a conformally covariant quantum field in four dimensions. It lives on the same Hilbert space as the AdS field, and shares the same unitary representation of $SO(2, 4)_0$. The limit changes the form of the AdS covariant transformation law of $\Phi(x, z)$ into the conformal transformation law of $\varphi(x)$.

A similar result was derived in a more abstract, purely algebraic setting [14]. This approach also allows to study "the way backwards from the boundary to the bulk", i.e., recovering the fields in AdS from their boundary limits. This proves to be a very subtle issue, and may fail in general. Whether the absence of a positive-definite Hilbert space in string theory and gauge theory may change the situation, because pertinent operator-algebraic NoGo theorems may not be valid, is not clear.

The above discussion shows two different relations between bulk and boundary fields: the boundary field as a limit $z \to 0$, and the dual boundary field defined by the fixed boundary values of the bulk path integral. The Maldacena conjecture refers to the latter, while the results in [12, 13, 14] refer to the former. How can these pertain to the same AdS-CFT relation? To exhibit the issue, let

$$Z(f) = (\Omega, e^{i\phi(f)}\Omega)$$

be the generating functional for the (Euclidean) correlation functions $(\Omega, \phi(x_1) \dots \phi(x_n)\Omega)$, i.e., the latter can be obtained by variation of $Z(f)$ w.r.t. the function f.

The usual path integral constructs $Z(f)$ by the formula

$$Z(f) := \int D[\psi(x)]\, e^{-S[\psi]+i\int dx\psi(x)f(x)}$$

where the integral extends over "all classical configurations" ψ, and $S[\psi]$ is a classical action. Now, the limit prescription for the boundary field is the path integral over the bulk field, with f supported only on the boundary:

$$Z^{\rm limit}_{\rm CFT}(f) := \lim_{z\to 0}\int D[\Psi(x,z)]\, e^{-S_{\rm AdS}[\Psi]+i\int dx\Psi(x,z)N(z)f(x)},$$

while the dual prescription is instead

$$Z^{\rm dual}_{\rm CFT}(f) := \int D[\Psi(x,r)]\, e^{-S_{\rm AdS}[\Psi]}\,\delta\big[\lim_z \widetilde{N}(z)\Psi(x,z)-f(x)\big].$$

Most remarkably, these two path integrals coincide! Any path integral requires a choice of a propagator. On AdS, there are two choices with different asymptotic behaviour at infinity, comparable to Dirichlet and Neumann boundary conditions in flat space. Evaluating the dual prescription with the "Neumann" propagator gives the same result as the limit prescription with the "Dirichlet" propagator.

The argument is only formal (using ill-defined path integrals), but can be shown to be stable under perturbation theory [15]. It can be traced back to symmetry, i.e., properties of the $SO(2,4)$-invariant propagators.

The actual Maldacena conjecture involving string theory and gravity is much richer than these purely field-theoretical considerations, and it is not proven by them. In more modern applications, the AdS-CFT (or more generally gravity-gauge) correspondence is rather not regarded as a *conjecture* $A = B$, but as a *definition* $B := A$ of conformal field theories that do not refer to an action $S[\varphi]$ of the CFT (which may not even exist) but instead to an auxiliary action $S_{\rm AdS}$. It has become a most successful tool to extend the "space of theories".

3 CFT in two dimensions

Conformal QFT in 2D is quite special. Most of its special features have their origin in the fact that the Minkowski metric factorizes into "lightcone coordinates" $x^\pm = x^0 \pm x^1 = t \pm x$:

$$dx^2 = dt^2 - dx^2 = dx^+ \cdot dx^-.$$

It eventually leads to factorization properties of conformal quantum fields, that allow to reduce much of the analysis to one-dimensional problems (referring to either x^+ or x^-). This not least simplifies computations and allows for exact solutions.

It is customary to call $x^\pm = t \pm x$ "chiral coordinates". The name goes back to the fact that the two-dimensional massless Dirac equation separates into two spinors $\Pi_\pm\psi(t,x) = \psi_\pm(t \pm x)$, that depend on only one lightcone coordinate. Here $\Pi_\pm$ are the projections on the eigenvalues ± 1 of the matrix $\gamma^5 = \gamma^0\gamma^1$, that in four dimensions determines the chirality of the particles in the usual sense of the orientation of the spin relative to the momentum (helicity).

A second characteristic feature of two dimensions is that there is a Lorentz-invariant distinction between "left" and "right", which eventually leads to the possibility of new types of commutation relations ("braid group statistics") beyond the familiar Bose-Fermi alternative.

3.1 The Möbius group

Notice that Lorentz transformations $\Lambda = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$ act on the lightcone coordinates like scale transformations: $(\Lambda x)^\pm = e^{\pm\theta}x^\pm$. Clearly, translations in lightlike directions shift one lightcone coordinate and preserve the other.

Also the conformal inversion $I(t,x) = \frac{(-t,x)}{t^2-x^2}$ takes to $(-t \pm x)/(t^2 - x^2) = -1/x^\pm$, and

$$C_b(x^\pm) = \frac{x^\pm}{1 - b^\pm x^\pm} \qquad (b^\pm = b^0 \pm b^1).$$

Thus, the Poincaré, scale and special transformations act on $x^\pm$ separately like the **Möbius group** $\mathsf{M\ddot{o}b}$ of fractional linear transformations

$$g(x) = \frac{ax + b}{cx + d} \qquad (ad - bc = 1).$$

Their composition law is the same as the product of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, hence $\mathsf{M\ddot{o}b}$ is isomorphic to $SL(2,\mathbb{R})/\mathbb{Z}_2$, and $SO(2,2)_0 \simeq \mathsf{M\ddot{o}b} \times \mathsf{M\ddot{o}b}$.

As in 4D, the conformal transformations may be singular on Minkowski spacetime, and should be regarded as regular transformations of the Dirac manifold. The two-dimensional Dirac manifold is given by the projective null cone ($X^2 = \vec{y}^2 - \vec{z}^2$ modulo rescaling, where $\vec{y}, \vec{z} \in \mathbb{R}^2$). This can be written as $\mathbb{M}^{1,1} = (S^1 \times S^1)/\mathbb{Z}_2$ and can be parametrized by two angles (τ, ξ) such that $(\tau,\xi) \sim (\tau + \pi, \xi + \pi)$. The conformal structure is given by $d\tau^2 - d\xi^2$. The 2D Minkowski coordinates are

$$t = \frac{\sin\tau}{\cos\tau + \cos\xi}, \quad x = \frac{\sin\xi}{\cos\tau + \cos\xi}.$$

In terms of the coordinates $\xi^\pm = \tau \pm \xi$, the identification is modulo 2π, so in these coordinates, $\mathbb{M}^{1,1}$ is just a product $S^1_+ \times S^1_-$ of two "chiral" circles. The Minkowski chiral coordinates become

$$x^+ = t + x = \frac{\sin\tau + \sin\xi}{\cos\tau + \cos\xi} = \tan\frac{\tau+\xi}{2} = \tan\frac{\xi^+}{2}, \quad x^- = t - x = \tan\frac{\xi^-}{2}.$$

It is also convenient to introduce the complex coordinates

$$z^\pm = e^{i\xi^\pm} = \frac{1+ix^\pm}{1-ix^\pm} \in S^1 \subset \mathbb{C} \quad \Leftrightarrow \quad x^\pm = i\frac{1-z^\pm}{1+z^\pm}.$$

The map $x \mapsto z$ is called **Cayley-transformation**; its inverse $z \mapsto x$ is the **stereographic projection** of the circle onto the real axis. Notice that $p_\pm = \frac{1}{2}(p_0 \pm p_1)$ are the generators of translations of $x^\pm$, and $k_\pm = \frac{1}{2}(k_0 \pm k_1)$ those of the special Möbius transformations $x^\pm \mapsto x^\pm/(1-b^\pm x^\pm)$. The generator $\ell_0 = \frac{1}{2}(p_0 + k_0)$ is the sum of $\ell_{0,\pm} = \frac{1}{2}(p_\pm + k_\pm)$, and $\ell_{0,\pm}$ are the generators of the shifts of the chiral angles $\varphi^\pm$ = rotations of the complex variables $z^\pm \in S^1$. In particular, $\exp(\pi\ell_{0,\pm})$ rotates by π, hence $z \mapsto -z$, hence $x \mapsto -1/x$ is the chiral inversion I.

3.2 Chiral fields and non-chiral fields

We have already seen that the conformal group $\mathrm{Conf}(2) = \mathsf{M\ddot{o}b} \times \mathsf{M\ddot{o}b}$ also factorizes into two Möbius groups acting (by fractional linear transformations) independently on the two chiral coordinates $x^\pm$. In QFT, the Lie algebra generators are given by i times self-adjoint operators P_μ, M_{01}, D, K_μ. The chiral generators $P_\pm = \frac{1}{2}(P_0 \pm P_1)$, $D_\pm = \frac{1}{2}(D \pm M_{01})$, $K_\pm = \frac{1}{2}(K_0 \mp K_1)$ commute for different chiralities, and for each chirality satisfy (index suppressed)

$$i[D,P] = P, \qquad i[D,K] = -K, \qquad i[P,K] = -2D.$$

Another feature of 2D is that the angle-preserving group is in fact much larger than $SO(2,2)_0 = \mathsf{M\ddot{o}b} \times \mathsf{M\ddot{o}b}$ because *any* pair of functions $x^+ \mapsto g^+(x^+)$, $x^- \mapsto g^-(x^-)$ rescales the metric by a factor (namely, $g^{+\prime}(x^+) \cdot g^{-\prime}(x^-)$). $g^\pm$ must be monotonously increasing (orientation preserving) maps of the circle on itself, i.e., the symmetry group is $\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$ (where smoothness is assumed). Clearly, $\mathsf{M\ddot{o}b} \subset \mathrm{Diff}_+(S^1)$, and we shall see that $\mathsf{M\ddot{o}b}$ is the maximal subgroup under which the vacuum may be invariant. Yet, this infinite-dimensional symmetry group allows to extract much more *a priori* results, cf Prop. 3.1 and Prop. 3.2.

Using $x^\pm$ as coordinates for fields $\phi(x^+, x^-)$, the infinitesimal transformation laws are fixed by a pair of two chiral scaling dimensions $h_\pm$ such that $h_+ + h_- = d$ (the proper scaling dimension under the dilations generated by $D = D_+ + D_-$) and $h_+ - h_- = s$ (the "spin", rather helicity, under the boost $M_{01} = D_+ - D_1$). The chiral generators transform only the dependence on the corresponding chiral coordinate. Suppressing the respective other chiral coordinate, the transformation laws become

$$i[P,\phi(x)] = \partial_x\phi(x), \quad i[D,\phi(x)] = (x\partial_x + d)\phi(x), \quad i[K,\phi(x)] = (x^2\partial_x + 2xd)\phi(x).$$

These transformation laws along with the invariance of the vacuum constrain the correlation functions to be of the form[7]

$$(\Omega, \varphi_1(x_1)\dots\varphi(x_n)\Omega) = \prod_{i<j} \frac{f(u_I^+, u_I^-)}{(x_{ij}^+)^{\mu_{ij}^+}(x_{ij}^-)^{\mu_{ij}^-}}$$

where the exponents $\mu_{ij}^\pm = \mu_{ji}^\pm$ satisfy $\sum_{j:j\neq i}\mu_{ij}^\pm = h_i^\pm$, and f is a function of the cross-ratios $u_I = \frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}$ (for either chirality, $x_{ij} = x_i - x_j \in \mathbb{R}$).

We have already mentioned the massless Dirac field decomposing in two components that depend only on $x^\pm = t \pm x$. Such fields are called "chiral fields". Chiral fields also arise whenever there are conserved (traceles symmetric) tensor fields, hence their presence is related to symmetries. The most important example is the stress-energy tensor:

Proposition 3.1 Lüscher-Mack theorem. *The stress-energy tensor of a 2D conformal QFT decomposes into two chiral fields $T_+(x^+)$ and $T_-(x^-)$ such that T_+ commutes with T_-, and each of them has commutation relations (suppressing the index)*

$$i[T(x), T(y)] = -(T(x) + T(y)) \cdot \delta'(x - y) + \frac{c}{24\pi} \cdot \delta'''(x - y)\,\mathbf{1}$$

with model-dependent non-negative constants c ($= c_\pm$) called "central charge".

By definition, a conformal stress-energy tensor is a covariant conserved and symmetric tensor field $T_{\mu\nu}$ such that

$$P_\mu = \int dx\, T_{0\mu}(t, x), \quad M_{\mu\nu} = \int dx\,(x_\mu T_{0\nu} - x_\nu T_{0\mu})(t, x),$$
$$D = \int dx\, x^\mu T_{0\mu}(t, x), \quad K_\mu = \int dx\,(2x_\mu x^\nu T_{0\nu} - x^2 T_{0\mu})(t, x)$$

are the t-independent generators of the conformal group. The Lorentz generators are independent of t because the stress-energy tensor is conserved: $\partial^\mu T_{\mu\nu} = 0$ and symmetric: $T_{\mu\nu} = T_{\nu\mu}$; D is independent of t iff the stress-energy tensor is traceless: $T^\mu_\mu = 0$, and this ensures that also K_μ are independent of t. Because $e^{isD}P_\mu e^{-isD} = e^s P_\mu$, P_μ has scaling dimension 1, and hence $T_{\mu\nu}$ must have scaling dimension 2.

Proof of Prop. 3.1: Because $T_{\mu\nu}$ is traceless and conserved, it has only two independent components $T_{00} = T_{11}$ and $T_{01} = T_{10}$. The conservation of T implies

$$(\partial_0 + \partial_1)(T_{00} - T_{01}) = 0, \qquad (\partial_0 - \partial_1)(T_{00} + T_{01}) = 0,$$

so that $T_\pm := T_{00} \pm T_{01}$ depend on one chiral coordinate $x^\pm$ only:

$$T_+ = T_+(x^+), \qquad T_- = T_-(x^-).$$

[7] Here, inverse powers $(x_{ij})^{-\mu}$ $(i < j)$ are understood as distributions $\lim_{\varepsilon\searrow 0}(x_i - x_j - i\varepsilon)^{-\mu}$, cf footnote 5.

$T_+(x^+)$ commutes with $T_-(y^-)$ because the point x can be shifted in the $-$-direction (without changing the operator $T(x_+)$) until it is spacelike from y, where the two operators commute by locality. For the commutator of $T_+(x^+)$ with $T_+(y^+)$, the same argument applies provided $x^+ \neq y^+$ (and similar for the other chirality). Hence, for either chirality (suppressing the index)

$$i[T(x),T(y)] = A_0(y)\delta(x-y) + A_1(y)\delta'(x-y) + \dots + A_3(y)\delta^{(3)}(x-y).$$

The sum stops at A_3 because T has scaling dimension 2, and hence A_n has scaling dimension $3-n$. This can be seen by applying the dilations $e^{isD}\cdots e^{-isD}$ to the ansatz. In particular, A_3 must be a multiple of $\mathbf{1}$ that is customarily called $c/24\pi$, and c is called "central charge".

One can determine the fields A_0, A_1, A_2, using the above infinitesimal transformation laws together with the relations

$$P = \int dx\, T(x), \quad D = \int dx\, xT(x), \quad K = \int dx\, x^2 T(x).$$

Thus, by taking integrals $\int dx$ over the ansatz, multiplied by x^n $(n = 0,1,2)$, one can compute $A_n(y)$, and gets the claimed commutator formula.

Because the 2-pt fn of T is $N\cdot(x-y-i\varepsilon)^{-4}$, the vacuum expectation of $i[T(x),T(y)]$ is $N\cdot i((x-y-i\varepsilon)^{-4} - (x-y+i\varepsilon)^{-4}) = -N\cdot \frac{i}{6}\partial_x^3((x-y-i\varepsilon)^{-1} - (x-y+i\varepsilon)^{-1}) = \frac{2\pi N}{6}\,\delta'''(x-y)$. By comparison, $c = 8\pi^2 N$. By positivity, N must be positive, hence $c > 0$. (If $c = 0$, then $T(x)\Omega = 0$, and by the Reeh-Schlieder theorem, $T(x) = 0$.) $\square$

A similar argument holds for commutators $i[T_\pm(x^\pm), \phi(y^+, y^-)]$: By locality is must be a sum of derivatives of $\delta(x-y)$ (the other chiral coordinate suppressed) with fields as coefficients, and the integrals with x^n $(n = 0,1,2)$ fix the lowest contributions:

$$i[T(x), \phi(y)] = \phi'(y)\delta(x-y) - h\phi(y)\delta'(x-y) + \dots$$

There are finitely many additional terms "$+\dots$" involving higher derivatives of $\delta(x-y)$ multiplying fields of lower dimension. A field without "$+\dots$" is called **primary**. It is the field of lowest scaling dimension within a family of fields ("secondaries") coupled among each other by commutators with the stress-energy tensor.

Integrating with an arbitrary real test fn $f(x)$, one gets for primary fields

$$i[T(f), \phi(y)] = f(y)\phi'(y) + hf'(x)\phi(y).$$

This is the infinitesimal form of the transformation law under diffeomorphisms $x \mapsto \gamma(x)$ for $\gamma(x) \approx x + f(x)$

$$U(\gamma)\phi(x)U(\gamma)^* = \Big(\frac{d\gamma(x)}{dx}\Big)^h \cdot \phi(\gamma(x)).$$

Thus, the stress-energy tensor is the generator of the diffeomorphism symmetry, $U(\gamma) = e^{iT(f)}$.[8]

Now, we apply the Cayley transform, defining the field $\phi(z^+, z^-)$ on $S^1 \times S^1$ by treating the Cayley transform as a Möbius transformation with complex coeffients:

$$\phi(z^+, z^-) := (dx^+/dz^+)^{h_+}(dx^-/dz^-)^{h_-} \cdot \phi(x^+, x^-).$$

This field is in general only periodic if $h_\pm$ are integers, otherwise the field will live on a covering of the circles (hence of $\mathbb{M}^{1,1}$). For the chiral stress-energy tensor T_+ one has $h_+ = 2$, $h_- = 0$ (and converse for T_-). Thus, $T(z) = (dx/dz)^2\,T(x)$ are periodic on S^1, and have a Fourier decomposition

$$T(z) = -\frac{1}{2\pi}\sum\nolimits_{n\in\mathbb{Z}} L_n z^{-n-2},$$

such that (in the original coordinates)

$$L_n = \frac{1}{2}\int dx\,(1-ix)^{1-n}(1+ix)^{1+n}\,T(x),$$

and in particular

$$L_0 = \frac{1}{2}(P+K), \qquad L_{\pm 1} = \frac{1}{2}(P-K) \pm iD.$$

The commutation relations of Prop. 3.1 turn into the **Virasoro algebra**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{n+m,0},$$

which is a central extension (the c-term) of the Lie algebra of $\mathrm{Diff}_+(S^1)$.

Möbius invariance of the vacuum is equivalent to $L_n\Omega = 0$ for $n = 0, \pm 1$. We have $L_n^* = L_{-n}$ and $[L_0, L_n] = -nL_n$, hence L_n decreases eigenvalues of L_0 by n. L_0 is positive because P and $K = IPI$ are positive. Thus, $L_0\Omega = 0$ implies $L_n\Omega = 0$ for $n \geq 0$. In contrast, $L_{-n}\Omega \neq 0$ for $n > 1$ because $\|L_{-n}\Omega\|^2 = (\Omega, [L_n, L_{-n}]\Omega) = \frac{c}{12}\,n(n^2-1) \neq 0$. Thus, the vacuum vector Ω cannot be diffeomorphism invariant.

3.3 Representations

Now, let $\phi(x)$ be a primary field of chiral dimension h. The primary commutation relations $[T(y), \phi(x)]$ turn into

$$2i[L_n, \phi(x)] = \Big(\frac{1+ix}{1-ix}\Big)^n \cdot [(1+x^2)\phi'(x) + 2h(1-inx)\phi(x)].$$

[8] More precisely, a function f defines a one-parameter group of diffeomorphisms by $\partial_t\gamma_t(x) = f(\gamma_t(x))$, hence infinitesimally $\gamma_t(x) = x + tf(x)$. Then γ_t is implemented by $U(\gamma_t) = e^{itT(f)} = e^{iT(tf)}$.

Because the generator P is positive, the operator e^{-yP} is well-defined for $y > 0$. This means that one can extend the vector function[9] $\Psi(x) = \phi(x)\Omega = e^{iPx}\phi(0)\Omega$ to $\Psi(x+iy) := e^{iP(x+iy)}\phi(0)\Omega \in \mathcal{H}$. $\Psi(\zeta)$ is an analytic function in the complex upper halfplane $\mathbb{C}_+$; under the Cayley transform, it is an analytic extension from $z \in S^1$ to $|z| < 1$.

The commutation relations between L_n and P imply that

$$2iL_n\Psi(\zeta) = \Big(\frac{1+i\zeta}{1-i\zeta}\Big)^n \cdot [(1+\zeta^2)\Psi'(\zeta) + 2h(1-in\zeta)\Psi(\zeta)].$$

In particular, the vector $|h\rangle := \Psi(z)|_{z=i}$ is an eigenvector of L_0 that is annihilated by all L_n $(n > 0)$:

$$L_0|h\rangle = h|h\rangle, \qquad L_n|h\rangle = 0 \quad (n > 0);$$

in other words: a ground state (or "lowest weight state") of the Virasoro algebra. Thus, every primary field of chiral dimension h gives rise to a positive-energy representation of the Virasoro algebra with lowest weight h.

This is remarkable: Recall that the Virasoro algebra is rather a re-writing of the commutation relations of the conformal QFT generated by the chiral stress-energy tensor. Thus, primary fields provide inequivalent representations (with different spectrum of L_0) of the same field algebra.

What is even more remarkable, is that the lowest weight h labelling (and in fact determining) the representation is nontrivially quantized without any model input. The argument goes like this [17]:

Because $|h\rangle$ is a ground state, the excited states are spanned by $L_{-n_k}\dots L_{-n_1}|h\rangle$ with eigenvalues $h + n_1 + \dots + n_k$. By virtue of the commutation relations, the operators may be ordered such that $n_k \geq \dots \geq n_1 > 0$. One can compute the scalar product among these vectors as functions of c and h, just by using the Virasoro algebra. E.g., $\|L_{-2}|h\rangle\|^2 = \langle h|[L_2, L_{-2}]|h\rangle = \langle h|4L_0 + \frac{c}{2}|h\rangle = 4h + \frac{c}{2}$ (if $\||h\rangle\| = 1$).

Proposition 3.2 *Three cases may arise.*

1. *Either the scalar product is positive definite. This happens for $c > 1$ and $h > 0$, and for $c = 1$ and $h > 0$ but $h \neq (\frac{1}{2}n)^2$ $(n \in \mathbb{N})$.*
2. *Or it is indefinite, i.e., there are states of negative norm square. Such values of (c, h) do not give rise to a Hilbert space, and must be discarded. This happens for $c < 0$, and for most values of h when $0 \leq c \leq 1$.*

[9] In the literature one finds formulations as if the field $\phi(x)$, and not only the vector $\Psi(x)$ were defined at complex points $x + iy$. This is not quite correct.

3. *Or it is positive semi-definite. Thus, the excited states span a Hilbert space, but they are not linearly independent. This happens for $c = 1$ and $h = (\frac{1}{2}n)^2$ ($n \in \mathbb{N}$), as well as for a discrete set of values of $c \in [0, 1)$:*

$$c = c_n = 1 - \frac{6}{n(n+1)} \quad (n = 3, 4, \dots),$$

and for each of them, for a finite set of values of h:

$$h = h_{p,q}(c_n) = \frac{[np - (n+1)q]^2 - 1}{4n(n+1)} \quad \begin{pmatrix} 1 \leq p \leq n-1 \\ 1 \leq q \leq n \end{pmatrix}.$$

Along with Prop. 3.1, these a priori *quantizations of field parameters and repns are without precendent in QFT.*

E.g., for $n = 3$ ($c = \frac{1}{2}$), the admissible values are $h = 0$, $h = \frac{1}{2}$, and $h = \frac{1}{16}$, and they come with vanishing linear combinations of excited states

$$L_{-1}|0\rangle = 0, \quad (4L_{-2} - 3L_{-1}^2)|\tfrac{1}{2}\rangle = 0, \quad \text{and} \quad (3L_{-2} - 4L_{-1}^2)|\tfrac{1}{16}\rangle = 0.$$

The ground state with $h = 0$ is the vacuum state, corresponding to the trivial "primary field" $\phi(x) = \mathbf{1}$, $|0\rangle = \Omega$. The representation of the Virasoro algebra generated by this state is, of course, just the vacuum representation of the chiral stress-energy tensor with central charge $c = \frac{1}{2}$. Indeed, the stress-energy tensor of the chiral components of the free real massless Fermi field in 2D $\psi_\pm(x^\pm) = \Pi_\pm\psi(t, x)$ (cf Sect. 3; Majorana = Dirac plus reality condition), has $c = \frac{1}{2}$, see Sect. 3.4. It is naturally represented on the Fermi Fock space, and this repn splits into two subrepns with even and odd Fermi number. These can be identified with the repns of lowest weights $h = 0$ and $h = \frac{1}{2}$.

In contrast, the representation of lowest weight $h = \frac{1}{16}$ is a new repn that cannot be realized by free fields. Instead, it can be identified with a sector of the Lorentzian version of the 2D Ising model at the critical value of the coupling constant, where the lattice correlation length diverges. At critical points, the continuum limit of lattice models is well-defined, and they become Euclidean quantum field theories.

The primary field of scaling dimensions $h_+ = h_- = \frac{1}{16}$ corresponds in the Euclidean to the magnetization density of the critical Ising model, which has scaling dimension (= critical exponent) $d = h_+ + h_- = \frac{1}{8}$. Thus, $h = \frac{1}{16}$ belongs to a genuinely interacting QFT, without any perturbative construction, nor lattice approximation, nor input of a Lagrangean.

The above classification of repns of the Virasoro algebra includes infinitely many quantum field theory models with nontrivial interactions. Some of the models with $c < 1$ have been identified with critical points of other Lorentzian spin models; but most of them were not previously known.

3.4 Models

Realizations of these admissible values can be obtained along the following lines.

The simplest free field chiral model is the massless Majorana field (= Dirac field with a hermiticity condition). Under the chiral projections $P_\pm = \frac{1}{2}(\mathbf{1} + \gamma^0\gamma^1)$, it splits into two fields $\psi_\pm$ that, by virtue of the Dirac equation, depend on $x^\pm$ only. (This is the origin of the name "chiral"). These hermitean chiral Fermi fields satisfy $\{\psi(x), \psi(y)\} = 2\pi\delta(x-y)$ with 2-pt fn $(\Omega, \psi(x)\psi(y)\Omega) = \frac{-i}{x-y-i\varepsilon}$ (scaling dimension $h = \frac{1}{2}$). Its chiral stress-energy tensor is $T = -\frac{1}{8\pi} :\psi \overset{\leftrightarrow}{\partial} \psi:$ with $c = \frac{1}{2}$.

Two hermitean Fermi fields combine into a complex Fermi field $\psi = \psi_1 + i\psi_2$ with stress-energy tensor $T = -\frac{1}{8\pi} :\psi^* \overset{\leftrightarrow}{\partial} \psi:$ with $c = 1$. One can construct the chiral current (the generator of the complex phase transformation) $j = \frac{1}{2}:\psi^*\psi:$ with 2-pt fn $(\Omega, j(x)j(y)\Omega) = \frac{-1}{(x-y-i\varepsilon)^2}$ (scaling dimension $h = 1$). Most remarkably, the current is again a free field: all its correlation functions coincide with those of the derivative $j_\pm = (\partial_0 \pm \partial_1)\varphi$ of the massless scalar Klein-Gordon field. It is a Bose field satisfying $[j(x), j(y)] = 2\pi i\delta'(x-y)$, and the complex Fermi stress-energy tensor can also be written as $T = \frac{1}{4\pi} :j^2:$. This phenomenon is known as "**bosonization**".

If a theory contains several primary currents j^a of dimension 1, their commutation relations are fixed by locality, conformal symmetry and the Jacobi identity:

$$-i[j^a(x), j^b(y)] = f^{ab}_c\, j^c(x) \cdot \delta(x-y) + \frac{k}{2\pi}\, g^{ab} \cdot \delta'(x-y)\, \mathbf{1},$$

where f^{ab}_c and g^{ab} are the structure constants and the (suitably normalized) Cartan metric of a simple Lie algebra $\mathfrak{g}$. This can be viewed as a non-abelian generalization of the above "abelian" current algebra $[j(x), j(y)] = 2\pi i\delta'(x-y)$. The parameter k is called the "level". Their stress-energy tensor is a multiple of $g_{ab}{:}j^aj^b{:}$ ("Sugawara construction"), and its central charge $c > 1$ is a rational function of k and the dimension and rank of the Lie algebra ($c = \frac{3k}{k+2}$ for $su(2)$).

The currents on the circle are defined via the Cayley transform: $j^a(z) := dx/dz \cdot j^a(x)$; under the Fourier decomposition

$$j^a(z) = \frac{1}{2\pi i} \sum\nolimits_{n\in\mathbb{Z}} j^a_n z^{-n-1},$$

the current algebra turns into the infinite-dimensional Lie algebra ("Kac-Moody algebra" or "affine Lie algebra")

$$[j^a_m, j^b_n] = if^{ab}_c j^c_{m+n} + kg^{ab}m\delta_{m+n,0} \cdot \mathbf{1}.$$

In particular, j^a_0 satisfy the commutation relations of the underlying Lie algebra $\mathfrak{g}$. The primary commutation relations with the stress-energy tensor turn into

$$[L_m, j^a_n] = -nj^a_{m+n}.$$

Thus, j^a_n $(n > 0)$ lower the eigenvalue of L_0, hence ground states must be annihilated by them. Because j^a_0 fix the eigenvalue, the ground states (and likewise all other eigenspaces of L_0) form a unitary representation $\underline{\lambda}$ of $\mathfrak{g}$. Selecting the ground state repn $\underline{\lambda}$ fixes the repn of the affine Lie algebra on the excited states. In analogy to the Virasoro case, one can compute scalar products among the states obtained by applying j^a_{-n} $(n > 0)$ to the ground state multiplets, and impose semi-definiteness. One finds that the level k must be a positive integer, and that not all unirepns $\underline{\lambda}$ of $\mathfrak{g}$ are allowed.

Recall that unirepns of simple Lie algebras are characterized by their "weight diagrams" = sets of joint eigenvalues of the Cartan subalgebra (= maximal system of commuting generators). For $su(2)$, the CSA is given by L_3 with eigenvalues = weights $m \in \frac{1}{2}\mathbb{Z}$, and a weight diagram is the integer-spaced set $m = -j, \dots, +j$; for $su(3)$ the Cartan generators are the diagonal matrices λ_3 and λ_8, and the weight diagrams are the triplet, octet, dekuplet etc diagrams familiar from flavour symmetry and QCD. Each irreducible unirepn has a unique "highest" weight. The set of all highest weights form a cone in the weight lattice, called the "Weyl alcove" (= the set $j = 0, \frac{1}{2}, 1, \dots$ for $su(2)$).

One finds that the allowed ground state repns $\underline{\lambda}$ of the affine Lie algebras have highest weight in a "truncated Weyl alcove" depending on the level (= the set of $j = 0, \frac{1}{2}, \dots \frac{k}{2}$ for $su(2)$, see the exercise). Thus, one has again an apriori quantization of the free parameter of the field algebra (here: the level k), and of the representations of the algebra.

Exercise: For $\mathfrak{g} = su(2)$, let $j^\pm_n = j^1_n \pm i j^2_n$ such that $[j^+_1, j^-_{-1}] = -2j^3_0 + k$ and $[j^3_0, j^\pm_n] = \pm j^\pm_n$, and let $|\ell m\rangle$ be a ground state with eigenvalues m of j^3_0 and $\ell(\ell+1)$ of $\vec{j}_0{}^2$. Compute recursively the norm square of $(j^-_{-1})^n|\ell m\rangle$ and conclude that $m \leq \frac{k}{2}$, hence $\ell \leq \frac{k}{2}$, and k must be a non-negative integer. (The case $k = 0$, hence $\ell = m = 0$ would imply that all vectors $j^a_{-n}|0\rangle$ vanish.)

The non-abelian current algebras with compact simple Lie algebras and level $k = 1$ can be constructed in terms of multiplets of free Fermi fields, schematically $j^a = {:}\psi^*\tau^a\psi{:}$. Higher levels can be constructed by the construction $j^a \otimes \mathbf{1} + \mathbf{1} \otimes j^a$ under which the level is additive. These models are thus very close to free fields – but already their representations with nontrivial $\underline{\lambda}$ do not occur in the free field Fock spaces, and the associated primary fields are interacting quantum fields, see below.

More interesting models can be obtained by the "coset construction": Given a chiral theory B and a subtheory A whose fields transforms correctly under the stress-energy tensor of B (e.g., the current algebras for a Lie algebra and a Lie subalgebra), one may consider the fields from B that commute with all fields of A. They form another subtheory C of B. Its stress-energy tensor is $T_C = T_B - T_A$ with central charge $c_C = c_B - c_A$. It is possible to choose current algebras A and B such that $c_C < 1$, and all admitted values $c_n = 1 - \frac{6}{n(n+1)}$ can be found [20]. Also for these, the representations with $h_{pq} \neq 0$ are not contained in any free field Fock space. How can the associated nontrivial primary quantum fields be constructed?

In general, their chiral scaling dimensions h_{pq} are not integers, so they cannot be chiral local fields. Instead, they must be fields with two nontrivial chiral dimensions $h^\pm$ such that the helicity $h^+ - h^-$ is integer. Hence, they have nontrivial commutation relations with two chiral stress-energy tensors $T_\pm(x^\pm)$.

Such fields are not free fields. Yet, one has a method to compute their correlation functions, that we illustrate for central charges $c^\pm < 1$. Because the scalar product of the excited states of the Virasoro algebra is only semi-definite (case 3 in Prop. 3.2), there are polynomials in L_{-n} that annihilate $|h\rangle$. E.g., for $c = \frac{1}{2}$ (Lorentzian Ising model), there are the null vectors $(4L_{-2} - 3L_{-1}^2)|\frac{1}{2}\rangle$ and $(3L_{-2} - 4L_{-1}^2)|\frac{1}{16}\rangle$. Hence $P(L_{-n})\phi(x)\Omega|_{x=i} = 0$. Using this equation in a correlation function, and commuting the polynomial through the other fields, gives a chiral differential equation for the correlation function ("Ward identity"), that has a finite-dimensional solution space [18]. E.g., the null vector $(3L_{-2} - 4L_{-1}^2)|\frac{1}{16}\rangle = 0$ turns into a lengthy partial differential equation

$$D(\partial_1, \partial_2, \partial_3)(\Omega, \phi(x_1)\dots\phi(x_4)\Omega)|_{x_4=i} = 0.$$

Inserting its cross-ratio representation as $\frac{f(u)}{(x_{12}x_{34})^{1/8}}$ (where $u = \frac{x_{12}x_{34}}{x_{13}x_{24}}$), one gets a (hypergeometric) ordinary differential equation for $f(u)$. It has two linearly independent solutions $f_{1,2}(u) = \sqrt{1 \pm \sqrt{1-u}}$. (In general, the solutions will be transcendental functions. They are called "conformal blocks", cf Sect. 2.5.)

The same argument applies for both chiralities, hence the 4-point function of the field $\varphi(x^+, x^-)$ with scaling dimensions $h^+ = h^- = \frac{1}{16}$ must be $(x^+_{12}x^+_{34})^{-\frac{1}{16}} \cdot (x^-_{12}x^-_{34})^{-\frac{1}{16}}$ $= (x^2_{12}x^2_{34})^{-\frac{1}{16}}$ times a quadratic expression in $f_{1,2}(u^+)$ and $f_{1,2}(u^-)$.

Locality of φ (symmetry under permutations of fields at spacelike distance) fixes this quadratic combination to be a multiple of $f_1 \cdot f_1 + f_2 \cdot f_2$, see the next section. All the higher correlation functions can be determined in a similar way [19]. In fact, the case $c = \frac{1}{2}$ is the Lorentzian version of the critical Ising model, and the field $\varphi(x^+, x^-)$ is the magnetization density.

This is a non-perturbative construction of a truely interacting quantum field.

3.5 Braid group statistics

The conformal block functions have branch cuts that are in conflict with symmetry under exchange $x_i^+ \leftrightarrow x_{i+1}^+$. Only in certain quadratic combinations

$$(\Omega, \phi(x_1, \dots, x_n)\Omega) = \sum\nolimits_{AB} N_{AB} f_A(x_i^+) f_B(x_i^-)/(\text{powers of } x_{ij}^\pm),$$

the cuts cancel each other in precisely the correct way to ensure locality of the correlation function under $(x_i^+, x_i^-) \leftrightarrow (x_{i+1}^+, x_{i+1}^-)$. The solutions (for each central charge c_n) with diagonal coefficient matrix N (in a suitable basis) are called "minimal models". Their n-point conformal blocks for every n have explicitly known integral representations, and the coefficients N_{AB} can be explicitly computed.

This gives an infinite family of interacting theories generalizing the Ising model.

An inspection of the precise form of the chiral conformal blocks $f_A(x_1, \dots, x_n)$ exhibits a new feature: under exchange of $x_i \leftrightarrow x_{i+1}$ they turn into linear combinations $R_i(\pm)_A^B f_B$ (times powers of cross-ratios, that absorb the change of the power factors), where the matrices $R_i(\pm)_A^B$ depends only on the sign of $x_i - x_{i+1}$.

This feature can be interpreted as commutation relations among chiral "exchange fields" [19]

$$\psi_a(x)\psi_b(y) = R(\pm)_{ab}^{cd} \psi_c(y)\psi_d(x).$$

The above quadratic expressions in terms of chiral conformal blocks are then just a consequence of the form of the local fields

$$\phi(x^+, x^-) = \sum n_{ab}\, \psi_a(x^+) \otimes \psi_b(x^-),$$

and the labels A of the conformal blocks are collective labels standing for a chain of indices a of n exchange field.

Since $x = (x^+, x^-)$ and $y = (y^+, y^-)$ are spacelike separated if either $x^+ > y^+$ and $x^- < y^-$, or vice versa, the commutativity is achieved by combining exchange matrices $R(+)$ with $R(-)$ for the two chiralities, that invert each other. At timelike separation, the combinations $R(+)$ with $R(+)$ fail to cancel, hence the Huygens principle does not hold for the non-chiral fields $\phi(x^+, x^-)$.

These commutation relations are called **braid group statistics**, because the matrices R_i for the exchanges in different positions $i, i + 1$ satisfy the defining relations of the braid group. These are the same as for the permutation group: $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ and $b_i b_j = b_j b_i$ if $|i - j| > 1$; but NOT $b_i = b_i^{-1}$.

The braid group statistics is a nontrivial generalization of **anyonic** statistics where $R(\pm) = e^{\pm i\alpha}$ with an arbitrary ("any") angle α. The exchange matrices of the minimal models yield previously unknown repns of the braid group, that turned out to provide a link between the the Jones polynomial (a topological invariant for knots regarded as closed braids) and Jones index (a famous invariant in the theory of vonNeumann algebras and noncommutative probability theory).

3.6 Diffeomorphism symmetry

We have seen that the stress-energy tensor is the generator of an infinite-dimensional symmetry of diffeomorphisms of S^1. This symmetry respects algebraic relations (commutators, etc), but the ground state is not invariant, cf Sect. 3.2.

The self-commutator of the stress-energy tensor in Prop. 3.1 is the infinitesimal form of the transformation law

$$U(\gamma)T(x)U(\gamma)^* = \Big(\frac{d\gamma(x)}{dx}\Big)^2 \cdot T(\gamma(x)) - \frac{c}{24\pi}\frac{D\gamma(x)}{Dx},$$

where $\frac{D\gamma(x)}{Dx} := \frac{\gamma'''}{\gamma'} - \frac{3}{2}(\frac{\gamma''}{\gamma'})^2$ is the Schwarzian derivative satisfying the cocycle relation $\frac{D\gamma_1\circ\gamma_2(x)}{Dx} = \gamma_2'(x)^2 \cdot \frac{D\gamma_1(\gamma_2(x))}{D\gamma_2(x)} + \frac{D\gamma_2(x)}{Dx}$. It vanishes exactly on the Möbius transformations: $\frac{D}{Dx}\frac{ax+b}{cx+d} = 0$. The presence of the Schwarzian derivative means that $T(x)$ is not a primary field.

(Exercise: Verify the composition law for $U(\gamma_1 \circ \gamma_2) = U(\gamma_1)U(\gamma_2)$, and show that for one-parameter groups γ_s as above, differentiation wrt s at $s = 0$ gives back the commutation relations of Prop. 3.1. These two properties ensure that infinitesimal diffeo's integrate to the finite transformation law.)

The above transformation law should actually be understood on S^1 (recall that $x^\pm \in \mathbb{R}$ is just a singular choice of coordinate on the Dirac manifold $S^1 \times S^1$ where conformal fields are defined without singularities). It reads exactly the same in terms of $\gamma(z) \in S^1$ and the field $T(z)$ as defined in Sect. 3.2, by virtue of the properties of the Schwarz derivative.

But the rhs of the transformation law also respects the commutation relations, when γ is not invertible, or defined only on open subsets of S^1. As an example, consider $\gamma_L(z) = \frac{L}{2\pi i} \cdot \log(z)$, that maps S^1 to a real interval of length L. On $\mathbb{R}$, this is $\gamma_L(x) = \frac{L}{\pi} \cdot \arctan x$. Hence one may *define* the stress-energy tensor on the interval I^L with periodic boundary conditions by the pullback

$$T^L(x) =: \Big(\frac{d\gamma_L^{-1}(x)}{dx}\Big)^2 \cdot T(\gamma_L^{-1}(x)) - \frac{c}{24\pi}\frac{D\gamma_L^{-1}(x)}{Dx}.$$

In contrast to diffeo's of S^1, this field is *not* unitarily equivalent to $T(x)$, but it satisfies the same commutation relations.

Defining both chiral fields $T^L{}_\pm(x^\pm)$ and $T^L_{00}(t,x) = T^L_{11}(t,x) := \frac{1}{2}(T^L_+(t+x) + T^L_-(t-x))$, $T^L_{01}(t,x) = T^L_{10}(t,x) := \frac{1}{2}(T^L_+(t+x) - T^L_-(t-x))$, one arrives at a traceless conserved stress-energy tensor on the "periodic strip" $\mathbb{R} \times I^L$. It describes a conformal QFT in a periodic box of size L with Hamiltonian $H^L = \int_{I^L} dx\, T^L_{00}(t,x)$. Inserting the definitions in terms of the originial chiral stress-energy tensors on S^1,

$$H^L = \frac{2\pi}{L} \cdot \Big(\frac{1}{2}(L_{0+} + L_{0-}) - \frac{c}{24}\Big).$$

The subtraction is due to the Schwarz derivative. Thus, the true Hamiltonian on the strip is the sum of the two conformal Hamiltonians shifted by $\frac{c}{24}$.

In a similar way, one can define conformal quantum field theories on many other "spacetimes" rather than $\mathbb{R} \times I_L$. They all share the same local structure, i.e., commutation relations involving δ functions in the chiral coordinates. Because the new fields are defined in terms of the original ones, they can be represented on the original Hilbert spaces; but they are in general not unitarily equivalent.

In String Theory, the strip is used as a paramatrization of the world sheet of a closed string. This is the reason why the conformal Hamiltonian shifted by $\frac{c}{24}$ is the relevant internal string Hamiltonian. In fact, in order to avoid an anomaly of the Poincaré symmetry, $\frac{c}{24}$ has to equal 1, hence $c \stackrel{!}{=} 24$. In the traditional canonical quantization of bosonic ST, the central charge is related to the spacetime dimension D of the target space where the string moves, and the condition translates into $D \stackrel{!}{=} 26$.

The above presentation is the Lorentzian version of what is found in most text books on CFT under the name "CFT on Riemann surfaces". Euclidean QFT is an analytic continuation of the correlation functions of Lorentzian QFT to "imaginary time", so that our chiral coordinates become complex variables $x^\pm = iy \pm x$, related by $x^- = -\overline{x^+}$. Under the Cayley transform, $z^- = \overline{z^+}$. Thus, chiral fields are "holomorphic" and "anti-holomorphic" in a single complex variable z; and by replacing local diffeomorphisms as above by holomorphic functions $\gamma(z)$, one arrives at Euclidean CFT on Riemann surfaces.

3.7 Characters

Consider the ground states $|h\rangle$ as in Sect. 3.3 and the representation spaces of the Virasoro algebra generated by them. The spectrum of L_0 is given by $h+n$ ($n \in \mathbb{N}_0$) with multiplicity given by the linearly independent states $L_{-n_k} \dots L_{-n_1}|h\rangle$ with total excitation number $n_1 + \cdots + n_k = n$. If no null vectors occur (e.g., if $c > 1$ and $h > 0$), then this is a simple combinatorial problem that is solved by the partition function

$$\mathrm{Tr}\, e^{-\beta L_0} = t^h \cdot p(t), \qquad p(t) \equiv \prod_{n=1}^\infty \frac{1}{1-t^n} \qquad (t = e^{-\beta}).$$

The coefficients of the power series $p(t)$ count the number of inequivalent partitions of a positive integer N into positive integers $n_k \geq \cdots \geq n_1$, each n_k corresponding to a ladder operator L_{-n_k}. Namely, the geometric series for each factor $(1-t^n)^{-1} = \sum_{r\geq 0} t^{rn}$ accounts for all states obtained by applying L_{-n} r times.

When there are null vectors ($c \leq 1$ or $h = 0$), then the combinatorial partition function $p(t)$ overcounts states, and there arise correction factors in the numerator. For $h = 0$, the correction factor is just $1 - t$, cancelling the geometric series for $n = 1$, thus taking care of the fact that $L_{-1}|0\rangle = 0$. For $c < 1$ and $h = h_{p,q} > 0$, the correction factors are explicitly nown, but more complicated [21].

One calls

$$\chi_h(t) := \mathrm{Tr}\,_h e^{-\beta(L_0-\frac{c}{24})}$$

in the representation generated by the lowest weight vector $|h\rangle$ the "characters" of the Virasoro algebra. In particular, the thermodynamic partition function of the CFT in the box (Sect. 3.6) is given by $\sum \chi_{h^+}(t)\cdot\chi_{h^-}(t)$, evaluated by $t = e^{-\frac{2\pi}{L\cdot k_BT}}$, where the sum extends over all pairs (h^+,h^-) such that the products of chiral ground states $|h^+\rangle\otimes|h^-\rangle$ occur in the Hilbert space. For the "minimal models" (Sect. 3.4), this is the sum over all $h^+ = h^- = h_{p,q}(c)$.

(For current algebra models, one defines modified characters $\chi(t,\underline{q}) = \mathrm{Tr}\, e^{-\beta(L_0-\frac{c}{24}-h_iQ_i)}$, where Q_i are the Cartan generators of the underlying Lie algebra measuring some internal quantum numbers ("isospin"), h_i are associated parameters like magnetic fields, and $q_i = e^{h_i}$.)

The most remarkable feature of the characters is a "high-temperature-low-temperature symmetry"called "modularity". Notice that the characters are easily evaluated at low temperature by just keeping a few leading terms of the power series in $t \ll 1$, while they clearly diverge at high temperature, $t \nearrow 1$. The modularity allows to quantify the divergence by an detailed asymptotic behaviour (and extract high-temperature thermodynamical equations of state).

One writes $t = e^{2\pi i\tau}$ and considers the "modular group" $SL(2,\mathbb{Z})/\mathbb{Z}_2$ of transformations

$$\tau \mapsto g(\tau) = \frac{k\tau+l}{m\tau+n}, \qquad (k,l,m,n\in\mathbb{Z},\; kn-mn=1),$$

that take the upper complex halfplane $\mathbb{C}_+$ to itself, hence $|t|<1$ to $|g(t)|<1$. It is generated by $T:\tau\mapsto\tau+1$ and $S:\tau\mapsto-\frac{1}{\tau}$, satisfying relations $S^2=(ST)^3=\mathrm{id}$. The former just takes $t^h\mapsto e^{2\pi ih}\cdot t^h$, hence

$$\chi_h(T(t)) = e^{2\pi i(h-\frac{c}{24})}\cdot\chi_h(t);$$

the latter takes $|t|\approx 1$ to $|S(t)|\approx 0$. In thermodynamic terms, $2\pi i\tau = -\beta$, hence $S(\beta) = 4\pi^2/\beta$ takes high to low temperature.

Modular symmetry of the characters is the statement that there exist unitary matrices (also called T and S) such that

$$\chi_a(T(t,\dots)) = T_{ab}\chi_b(t,\dots), \qquad \chi_a(S(t,\dots)) = S_{ab}\chi_b(t,\dots).$$

The index a runs over the lowest weight vectors of the model (i.e., the labels (p,q) in the minimal models, and over the "truncated Weyl alcove" of lowest weights of the underlying Lie algebra in current algebra models), and $\dots$ stands for possible "magnetic" parameters in the current algebra cases, that are also transformed by T and S. T is always the diagonal matrix with entries $e^{2\pi i(h-\frac{c}{24})}$.

These formulas are nontrivial generalizations of Ramanujan's inversion formula (famous in number theory) for the combinatorial partition function

$$t'^{-\frac{1}{24}}p(t') = (i/\tau)^{\frac{1}{2}} \cdot t^{-\frac{1}{24}}p(t)$$

where $t = e^{2\pi i\tau}$, $t' = S(t) = e^{-2\pi i/\tau}$. This formula can already be used to determine the high-temperature behaviour of the characters for $c > 1$.

In the general case with primary fields that are not hermitean, the matrices S and T actually represent $SL(2,\mathbb{Z})$ (and not only the quotient by $\mathbb{Z}_2$). They satisfy the relations

$$S^2 = (S^{-1}T)^3 = C$$

where C is the matrix $C_{ab} = \delta_{\overline{a}b}$, where $\overline{a}$ stands for the representation generated by the hermitean conjugate field. Obviously $C^2 = \mathbf{1}$. Moreover, the remarkable "Verlinde formula" holds [22]:

$$\sum_d \frac{S_{ad}S_{bd}S^*_{cd}}{S_{0d}} = N^c_{ab}.$$

Here, the sum runs over all (finitely many) representations, 0 stands for the vacuum representation ($h = 0$), and N^c_{ab} are integer "fusion coefficients" that count the number of copies of a field φ_c appearing in the operator product expansion of $\varphi_a \cdot \varphi_b$.

These remarkable properties were established explicitly for Virasoro and affine Lie algebras, and were shown to pass to their coset models. They could also be derived by a formal argument using path integrals, by the following idea: A partition function at inverse temperature β can be represented as a path-integral over a periodic interval β of imaginary time. If the system is in a periodic box of length 2π, one gets a torus with the sides β and 2π. But the path integral over a torus should be invariant under exchange of the two sides, hence the partition function should be the same for inverse temperatures β and $4\pi^2/\beta$. In a CFT in 2D, the total partition function is a sum of products of chiral characters, and the invariance of the total partition functions requires a linear transformation law for the chiral characters.

Accepting this formal argument, the modular transformation law of characters was elevated to a postulate, and used for classifications of unknown models [23]. The mere fact that the unitary matrix S produces non-negative integers in the Verlinde matrix, and satisfies the $SL(2,\mathbb{Z})$ relations with the diagonal matrix T, poses strong constraints on the entries of T, i.e., the on scaling dimensions of a theory. Unfortunately, it turned out that there exist solutions to the modular properties for which an associated CFT does not exist.

There exists an approach to QFT that exploits properties of vonNeumann algebras (of observables on a Hilbert space) rather than properties of fields. Almost everything we have learned about CFT can be reformulated in this framework. In this approach, "primary fields" are replaced by "representations of the chiral algebra" (we have already used this relation), and the fusion coefficients N_{ab}^c arise as multiplicities in the decomposition of a "product of representation", that shares many properties with the tensor product of representations of a group (although it is not a tensor product!).

One can formulate braid-group statistics (Sect. 3.5) very naturally in this framework as a unitary representation of the braid group ("statistics operators"); and it is a most intriguing fact that one can extract from this representation unitary matrices S and T (certain traces of simple statistics operators) that satisfy the $SL(2,\mathbb{Z})$ relations and the Verlinde formula [25, 24, 27], but the relation to modular transformations of characters can be established in general only for the matrix T. (This fact is a conformal Spin-Statistics theorem beyond the Fermi-Bose alternative that does not hold in two spacetime dimensions.) In cases where modular transformations under S are known, the modular and the braid group S matrices also coincide.

3.8 Boundary CFT

CFT in 2D admits an intersting explicit version of the AdS-CFT correspondence[26]. The standard chart of AdS_2 is the Minkowski halfspace $\mathbb{R}^{1,1}_+ = \{(t,x) : x > 0\}$ with the metric $ds^2 = \frac{1}{c^2}(dt^2 - x^2)$. Here, x is the same as the coordinate called z in Sect. 2.6, measuring the distance from the boundary. Thus, a QFT on AdS_2 is a QFT on the halfspace, with a boundary at $x = 0$. In the construction to be sketched, this QFT will also be conformal, hence it is called "boundary CFT". The sketch will be highly non-technical.

A point (t,x) in the halfspace is specified by the pair $x^\pm = t \pm x$ that may be viewed as "advanced and retarded" times on the time axis, $x^+ > x^-$. Now suppose a chiral CFT is given on the time axis $\mathbb{R}$, e.g., a stress-energy tensor with $c < 1$. For $(t,x) \in \mathbb{R}^{1,1}_+$, define

$$T_{00}(t,x) = T_{11}(t,x) = \frac{1}{2}(T(x^+)+T(x^-)), \quad T_{01}(t,x) = T_{10}(t,x) = \frac{1}{2}(T(x^+)-T(x^-)).$$

This is a stress-energy tensor on $\mathbb{R}^{1,1}_+$ constructed from a single chiral stress-energy tensor (unlike the situation in Prop. 3.1 where the 2D stress-energy tensor is composed of two opposite chiral stress-energy tensors). It is local on the halfspace, and satisfies the boundary condition $T_{01}(t,0) = 0$. Because T_{01} is the energy flow, the boundary condition means that no energy flows out of the boundary.

We say that an operator is "localized in an interval" if it is a chiral field smeared over the interval or any function of smeared fields. All operators localized in a given interval form an algebra.

For $(t,x) \in \mathbb{R}^{1,1}_+$, define intervals $I_{+\varepsilon} = (x^- - \varepsilon, x^+ + \varepsilon)$ and $I_{-\varepsilon} = (x^- + \varepsilon, x^+ - \varepsilon)$;

i.e., $I_{+\varepsilon}$ is slightly larger than $I_{-\varepsilon}$. Consider the algebra of operators localized in $I_{+\varepsilon}$ that commute with all operators localized in $I_{-\varepsilon}$. In the limit $\varepsilon \to 0$, these "relative commutants" contain local fields $\varphi(t,x)$ on the halfspace. The crucial feature of locality is an immediate consequence of the construction via relative commutants: If two points x, y are spacelike separated, and WLOG y to the right of x, then $I_{+\varepsilon}(x)$ is contained in $I_{-\varepsilon}(y)$ for sufficiently small ε, hence $\varphi(y)$ commutes with $\varphi(x)$.

Moreover, the stress-energy tensor $T_{\mu\nu}(x)$ provides generators of conformal transformations of the halfspace fields. Now consider only fields at a distance $x > R$ from the boundary, and take the limit $R \to \infty$. This is the same as "moving the boundary to the left". The resulting algebra is a CFT in 2D without boundary. One can even show that if the chiral theory is a stress-energy tensor with $c < 1$, then the resulting CFT is the associated minimal model, cf Sect. 3.4.

This constructs an interacting local CFT on $\mathbb{R}^{1,1}$ from a free chiral CFT on $\mathbb{R}$.

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