\[ \text{Ai}(x) = \frac{1}{2\pi i} \int_C \exp\left[ \frac{1}{3} t^3 - xt \right] dt \]
When does the steepest line for one saddle hit the other saddle?

**Definition**
The Stokes line is a line on \( x \)-plane which satisfies

\[
\text{Re } S(\xi_1(X)) = \text{Re } S(\xi_2(X))
\]

**Definition**
The turning point is a point which satisfies

\[
\frac{\partial S(\xi, X)}{\partial \xi} = \frac{\partial^2 S(\xi, X)}{\partial \xi^2} = 0
\]

Local analysis around a turning point tells us that three Stokes lines emanate from a turning point:
Two saddles collide at the turning point
Asymptotic expansion for Airy function

\[ Ai(z) \sim \frac{e^{-\xi}}{2\pi^{1/2}z^{1/4}} \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{\xi^n} \quad (|\arg z| < \pi) \]

where \( \xi = \frac{2}{3}z^{3/2} \) and

\[ c_0 = 0, \quad c_n = \frac{2^n}{3^{3n}} \frac{\Gamma(3n + 1/2)}{\Gamma(2n + 1)\Gamma(1/2)} \quad (c \geq 1) \]
$z = 3$

$z = 5$

$z = 7$
Asymptotic expansion for \( f(z) \)

\[
f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \quad (z \to \infty \text{ in } \arg z < \pi/2)
\]

Since for any \( n, z^n e^z = 0 \) as \( z \to \infty \)

\[
e^z \sim 0 + \frac{0}{z} + \frac{0}{z^2} + \cdots \quad (z \to \infty \text{ in } \arg z < \pi/2)
\]

which implies \( g(z) = f(z) + \text{const} \cdot e^z \) has the same asymptotic expansion as \( f(z) \).

Or equivalently, asymptotic expansions cannot resolve exponentially small quantities.
“When the cat’s away the mice may play. You are the cat and I am the mouse. I have been doing what I guess you won’t let me do when we are married, sitting up till 3 o’clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy’s, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it. I took it up again a few days ago, and after a two or three days’ fight, the last of which I sat up till 3, I at last mastered it. I don’t say you won’t let me work at such things, but you will keep me to more regular hours. A little out of the way now and then does not signify, but there should not be too much of it. It is not the mere sitting up but the hard thinking combined with it...”

Recent development of asymptotic analysis based on the Borel resummation

- **Resurgent analysis**  
  *Ecalle, Voros, Balian, Pham-Delabaere, Sauzin, ⋅⋅⋅*

- **Exponential asymptotics**  
  *Dingle, Berry-Howls, King, Daalhuis, Olver, ⋅⋅⋅*

- **Exact WKB analysis**  
  *Sato-Aoki-Kawai-Koike-Takei, ⋅⋅⋅*
Resurgent theory: a sketch

divergent series $\rightarrow$ Borel transform $\rightarrow$ (analytic continuation) $\rightarrow$ Laplace transform

singularities on the Borel plane

\[ \uparrow \]

Stokes phenomenon

Construction of global (WKB) solutions

Ⅱ

connection formula ( = local )

+ 

Riemann sheet structure of the Borel transform ( = global )
Borel resummation method — an example —

**Euler equation**

\[- \frac{df(z)}{dz} + f(z) = \frac{1}{z}\]

**Formal power series solution**

\[f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{z^{n+1}}\]

**Laplace transform**

\[\mathcal{L} \zeta^n \equiv \int_0^\infty \zeta^n e^{-z\zeta} d\zeta = \frac{n!}{z^{n+1}} \Rightarrow \mathcal{L}^{-1} \frac{n!}{z^{n+1}} = \zeta^n\]

**Borel transform (inverse Laplace transform) of \(f(x)\)**

\[f_B(\zeta) \equiv \mathcal{L}^{-1} f(z) = \mathcal{L}^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{z^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \zeta^n = \frac{1}{1 + \zeta}\]

**Borel sum of \(f(z)\)**

\[f(z) = \mathcal{L} f_B(\zeta) = \int_0^\infty f_B(\zeta)e^{-z\zeta} d\zeta = \int_0^\infty \frac{e^{-z\zeta}}{1 + \zeta} d\zeta\]
Analytical continuation and Stokes phenomenon

Analytical continuation of Borel sum of $f(z)$

$$f(z) = \int_0^{\infty e^{i\theta}} \exp(-z\zeta) f_B(\zeta) d\zeta$$

For $\theta = \pi - \delta$

$$f(z) = \int_{\Gamma} \frac{e^{-z\zeta}}{1 + \zeta} d\zeta$$

For $\theta = \pi + \delta$

$$f(z) = \int_{\Gamma_1} \frac{e^{-z\zeta}}{1 + \zeta} d\zeta + \int_{\Gamma_2} \frac{e^{-z\zeta}}{1 + \zeta} d\zeta$$

$$\int_0^{\infty e^{i\theta}} \frac{e^{-z\zeta}}{1 + \zeta} d\zeta + \frac{2\pi i e^z}{\Gamma}$$
**Much less trivial example — Cusp —**

\[ S(\xi; x, y) = \frac{1}{4} \xi^4 + \frac{1}{2} y \xi^2 + x \xi \]

"The Stokes set of the cusp diffraction catastrophe"


**Stokes set**

\[ \text{Re } S(\xi_1(X)) = \text{Re } S(\xi_2(X)) \]

**Bifurcation set (caustics)**

\[ \frac{\partial S(\xi, X)}{\partial \xi} = \frac{\partial^2 S(\xi, X)}{\partial \xi^2} = 0 \]
For each turning point $O_i$ ($i = 1, 2$), find the Stokes lines emanating from it, using the condition

$$\text{Re } S(\xi_\alpha(X)) = \text{Re } S(\xi_\beta(X)) \quad (\alpha \neq \beta)$$

Note, for example, $O_1$ is the turning point associated with $\xi_1$ and $\xi_2$, $O_2$ associated with $\xi_2$ and $\xi_3$, but why not between $\xi_1$ and $\xi_3$?

\[ \text{Re } S(\xi_1) = \text{Re } S(\xi_3) \]

\[ \text{Re } S(\xi_1) = \text{Re } S(\xi_2) \]

\[ \text{Re } S(\xi_2) = \text{Re } S(\xi_3) \]
Find the turning points using the condition \( \frac{\partial S(\xi, X)}{\partial \xi} = \frac{\partial^2 S(\xi, X)}{\partial \xi^2} = 0 \)

where \( S(\xi; x, y) = \frac{1}{4} \xi^4 + \frac{1}{2} y \xi^2 + x \xi \)

For a fixed \( x = x_0 \), we have two turning points on y-plane:

\( O_1 = (x_0, y_1) \)

\( O_2 = (x_0, y_2) \)

Re \( y \), Im \( y \), x = 5.0

\( \theta = 0 \)

\( \theta = \pi \)

\( \theta = \frac{\pi}{3} \)

\( \theta = \frac{2\pi}{3} \)

\( \theta = \frac{5\pi}{6} \)

\( \theta = \pi \)

Outline

1. Saddle point method and the Stokes phenomenon
2. The cases with more than two saddles
3. Virtual turning and new Stokes curves
4. An idea of pruning theory of the Stokes geometry
5. Cancellation of saddle point contributions due to global correlation

Pruning theory of the Stokes geometry
Akira Shudo
Department of Physics, Tokyo Metropolitan University
Why does the Stokes connection occur on one side, but not on the other side?

\[ I(k, X) = \sum_i \int_{C_i} d\xi \exp[ikS(\xi, X)] \]
\[ = \sum_i I_i(k, X) \]

\( C_i \): steepest descent line passing through the saddle \( \xi_i \)

Univaluedness not satisfied ...
Decompose the integral into a sum over saddles

Why does the Stokes connection occur on one side, but not on the other side?

\[ I(k, X) = \sum_i \int_{C_i} d\xi \exp[ikS(\xi, X)] \]

\[ = \sum_i I_i(k, X) \]

\[ C_i : \text{steepest descent line passing through the saddle } \xi_i \]

\[ I_3 \rightarrow I_3 \]

\[ \rightarrow I_3 + \sigma_1 I_1 \]

\[ \rightarrow I_3 + \sigma_2 I_2 + \sigma_1 I_1 \]

\[ \text{Univaluedness recovered if } \]

\[ \sigma_1 \sigma_2 = \sigma_1' \]
From where does the new Stokes curve emanate?
— Ordinary turning point case —

Two saddles collide at the turning point

\[
\text{Re } S(\xi_1) = \text{Re } S(\xi_2) \quad \text{Im } S(\xi_1) = \text{Im } S(\xi_2)
\]
From where does the new Stokes curve emanate?

Virtual turning point (Aoki-Kawai-Takei 1992)

\[
\Re S(\xi_1) = \Re S(\xi_3) \quad \Im S(\xi_1) = \Im S(\xi_3)
\]
The virtual turning point as the self-intersection point of bicharacteristic curves

Definition (Aoki-Kawai-Takei) The virtual turning point is a self-intersection point of the projection of a bicharacteristic strip (= solution curves of the bicharacteristic equation) onto \((x, y)\) plane.

- ordinary turning points:
  \[ x'(t) = y'(t) = 0 \text{ at } t = t_0 \]

- virtual turning points:
  \[ x(t) = x(s) \text{ and } y(s) = y(t) \text{ at } t \neq s \]
Higher-order differential equations and its Borel transform

$m$-th order differential eq.

\[ H\psi = 0 \quad \text{where} \quad H = \sum_{0 \leq i \leq m, 0 \leq j \leq n} a_{ij} x^i k^{m-i} \left( \frac{d}{dx} \right)^i \]

Its Borel transform

\[ H_B\psi_B = 0 \quad \text{where} \quad H_B = \sum_{0 \leq i \leq m, 0 \leq j \leq n} a_{ij} \frac{\partial^{m-j}}{\partial y^{m-j}} \left( -\frac{\partial}{\partial \xi} \right)^j \xi^i \]

where \( \psi_B \) is the Borel transform of \( \psi \).

Let \( P_B \) the principal symbol for the operator \( H_B \). Then the bicharacteristic equations for \( H_B \) ( = Hamilton-Jacobi equation ) are written as

\[
\begin{align*}
\frac{dx}{dt} &= \frac{\partial P_B}{\partial \xi}, & \frac{d\xi}{dt} &= -\frac{\partial P_B}{\partial x}, & \frac{dy}{dt} &= \frac{\partial P_B}{\partial k}, & \frac{dk}{dt} &= -\frac{\partial P_B}{\partial y}, & P_B(x, \xi, \eta) &= 0
\end{align*}
\]
A recipe for finding a complete Stokes geometry

step1  Find all the ordinary turning points.

step2  Draw the ordinary Stokes lines emanating from them.

step3  Find intersection points between ordinary Stokes lines.

step4  For each intersection point, find a virtual turning point.

step5  Draw new Stokes lines emanating each virtual turning point.

step6  Determine the nature of connection around each crossing point by checking the univaluedness condition.

step7  Find intersection points between Stokes lines emanating from 1-st order and those from 0-th order turning points.

step8  Repeat this procedure until relevant Stokes lines do not appear.
Model for nonadiabatic transition

Time-dependent 2-level problem (Landau-Zener)

\[i\hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \rho_1(t) & c_{12} \\ \frac{c_{12}}{c_{12}} & \rho_2(t) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\]

\[\rho_1(t) = b_1 t + a, \quad \rho_2(t) = b_2 t, \quad c_{12} = \text{const.}\]

Time-dependent multi-level problem (Aoki-Kawai-Takei)

\[i\hbar \frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \rho_1(t) & c_{12} & c_{13} \\ \frac{c_{12}}{c_{31}} & \rho_2(t) & c_{23} \\ \frac{c_{13}}{c_{32}} & \frac{c_{23}}{c_{32}} & \rho_3(t) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}\]

i) \[\rho_1(t) = b_1 t + a, \quad \rho_2(t) = b_2 t, \quad \rho_3(t) = b_3 t\]

ii) \[\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c\]
ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$
Possible situations where new Stokes play a role

**Case 1)** We can avoid crossing new Stokes curves, but need information on new Stokes curves and virtual turning points in order to find a connection path which does not pass any new Stokes curves.

**Case 2)** We cannot avoid crossing new Stokes curves because of the presence of a chain of active new Stokes curves.
ii) \( \rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c \)

\[
c = 0.0
\]

\((-2.0000000000e+00,-2.0000000000e+00)- (2.0000000000e+00, 2.0000000000e+00)\)
\[ \rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c \]

\[ c = 0.1 \]

\((-2.0000000000e+00, -2.0000000000e+00)\) \(- (2.0000000000e+00, 2.0000000000e+00)\)
ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$

$c = 0.4$
ii) \( \rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c \)

\( c = 0.7 \)
ii) \( \rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c \)

\[ c = 0.9 \]

\((-2.0000000000e+00,-2.0000000000e+00)-(2.0000000000e+00,2.0000000000e+00)\)
ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$

$c = 1.2$

$(-2.0000000000e+00, -2.0000000000e+00) - (2.0000000000e+00, 2.0000000000e+00)$