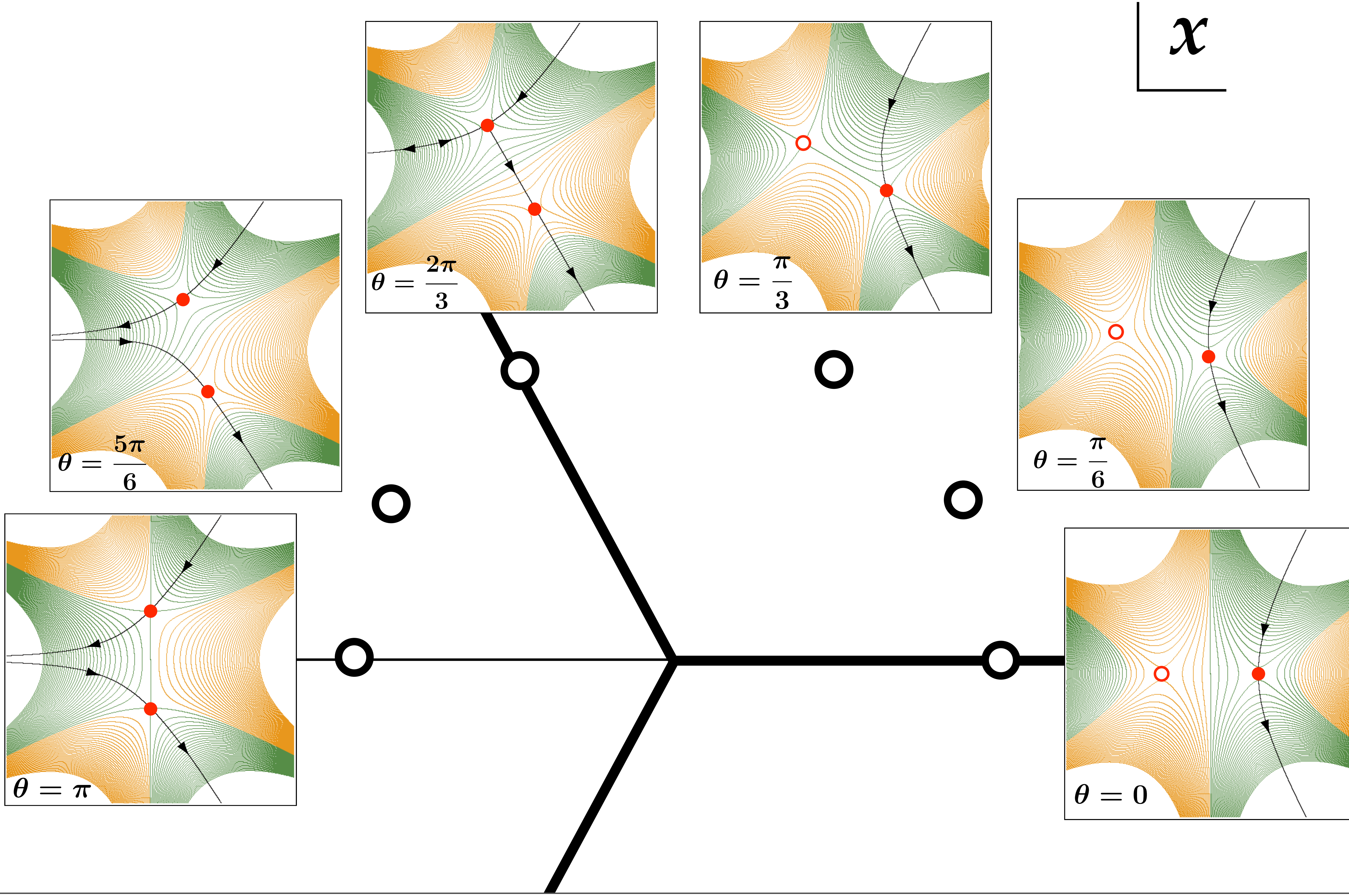
$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_C \exp\left[\frac{1}{3}t^3 - xt\right]dt$$



When does the steepest line for one saddle hit the other saddle ?

Definition

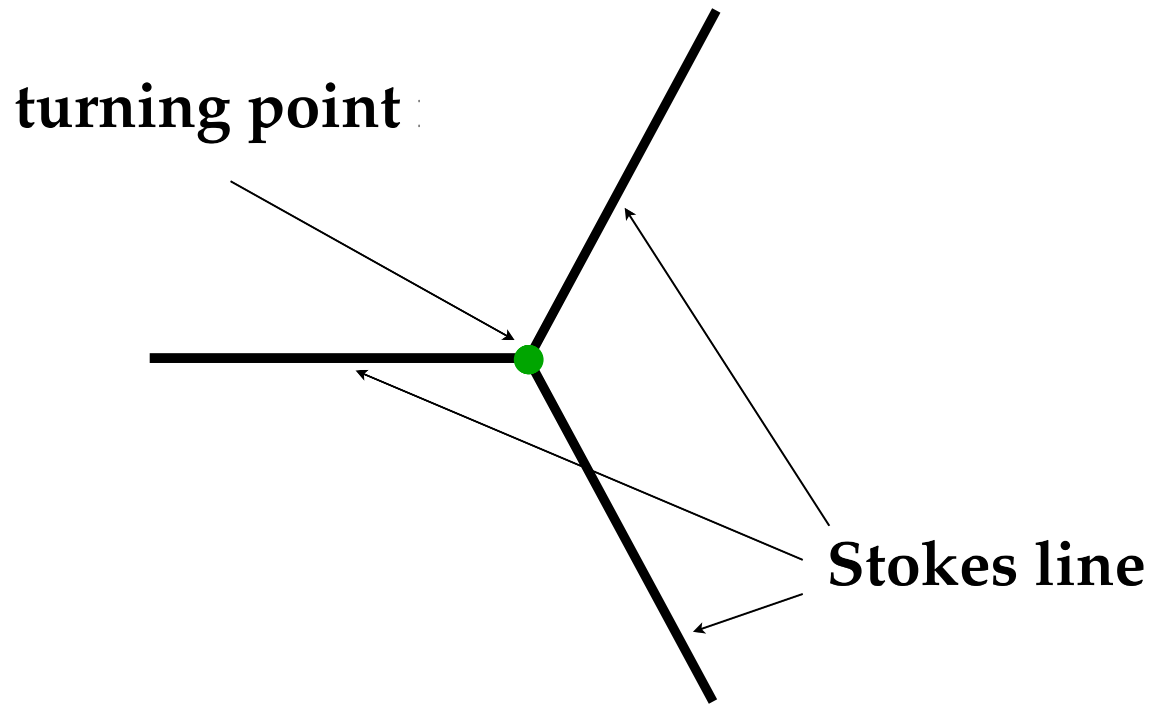
The Stokes line is a line on x-plane which satisfies

$$\mathrm{Re}\, S(\xi_1(X)) = \mathrm{Re}\, S(\xi_2(X))$$

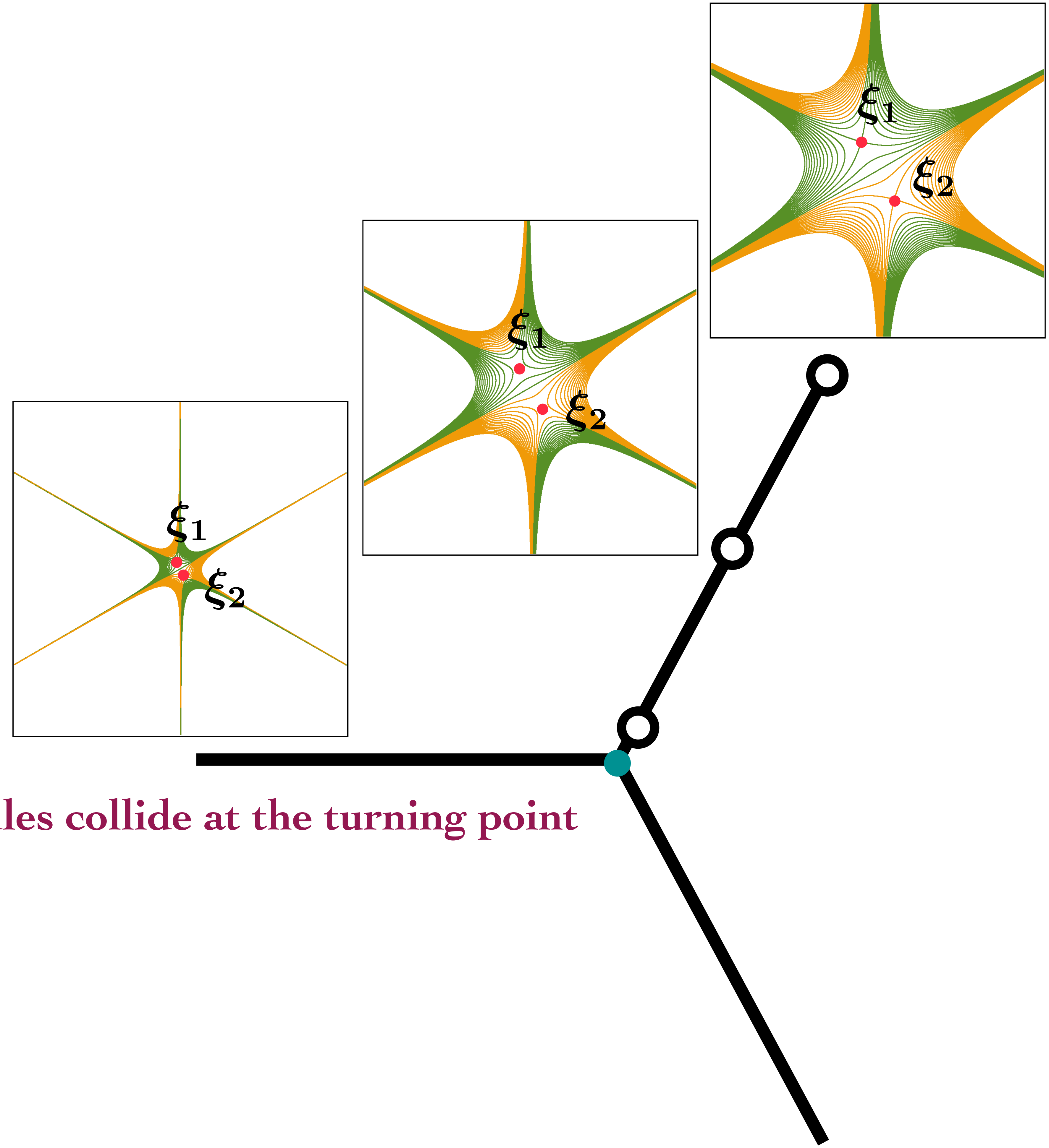
Definition

The turning point is a point which satisfies

$$\frac{\partial S(\xi, X)}{\partial \xi} = \frac{\partial^2 S(\xi, X)}{\partial \xi^2} = 0$$



Local analysis around a turning point tells us that three Stokes lines emanate from a turning point:



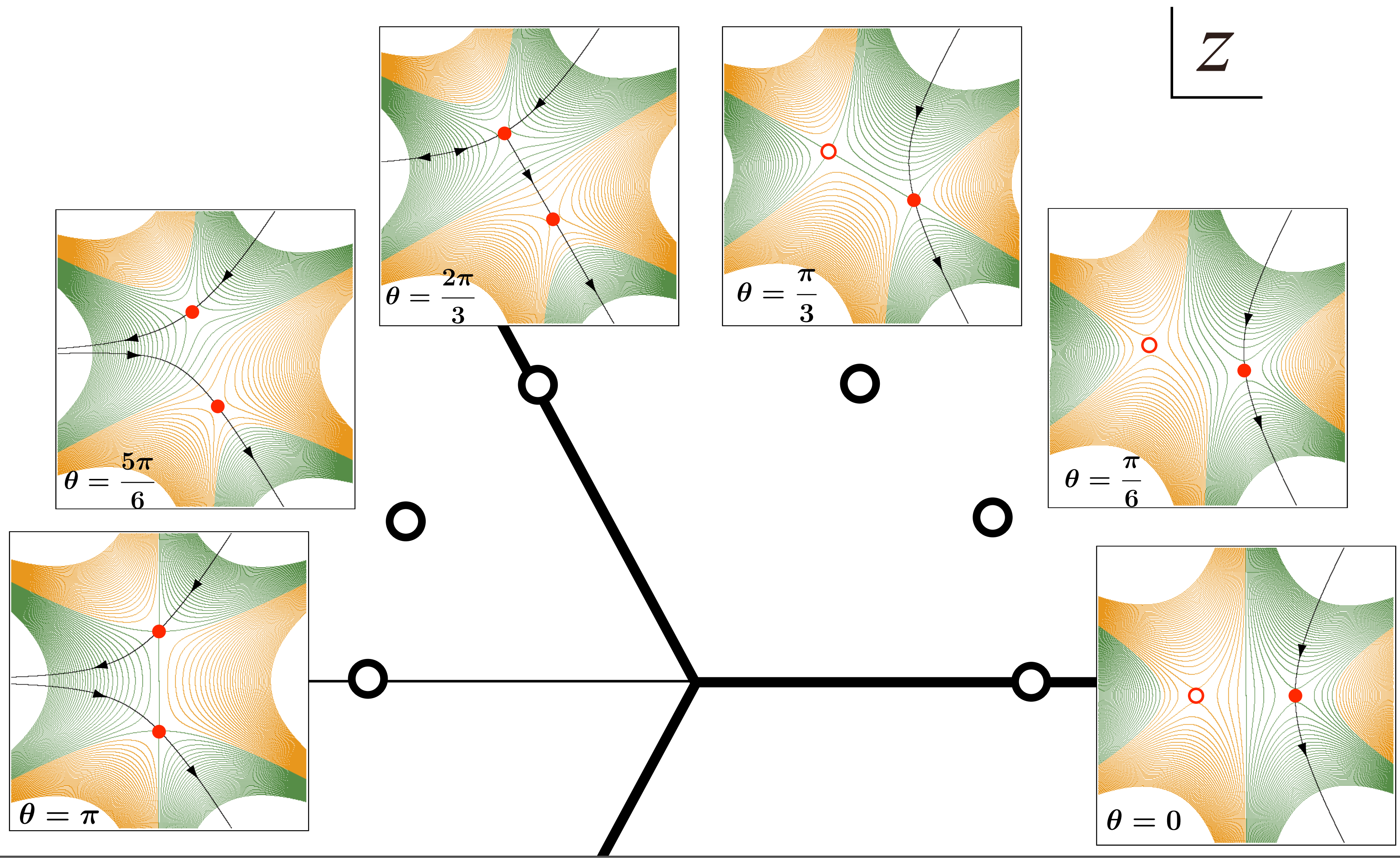
Two saddles collide at the turning point

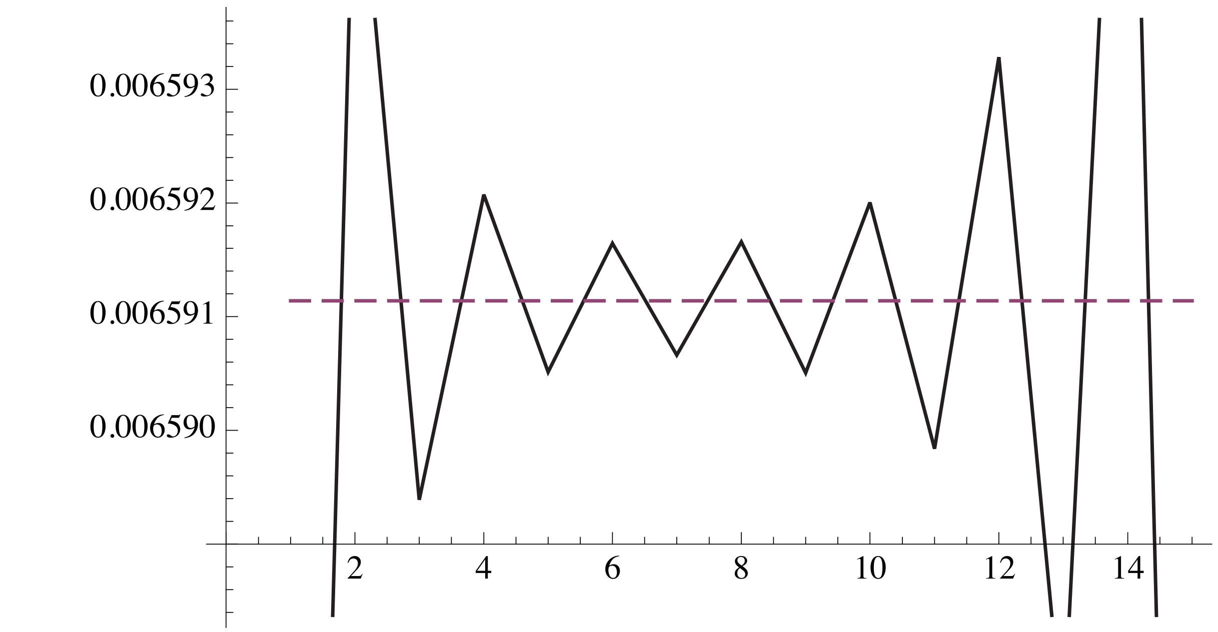
Asymptotic expansion for Airy function

$$\mathrm{Ai}(z) \sim \frac{e^{-\xi}}{2\pi^{1/2} z^{1/4}} \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{\xi^n} \qquad (|\arg z| < \pi)$$

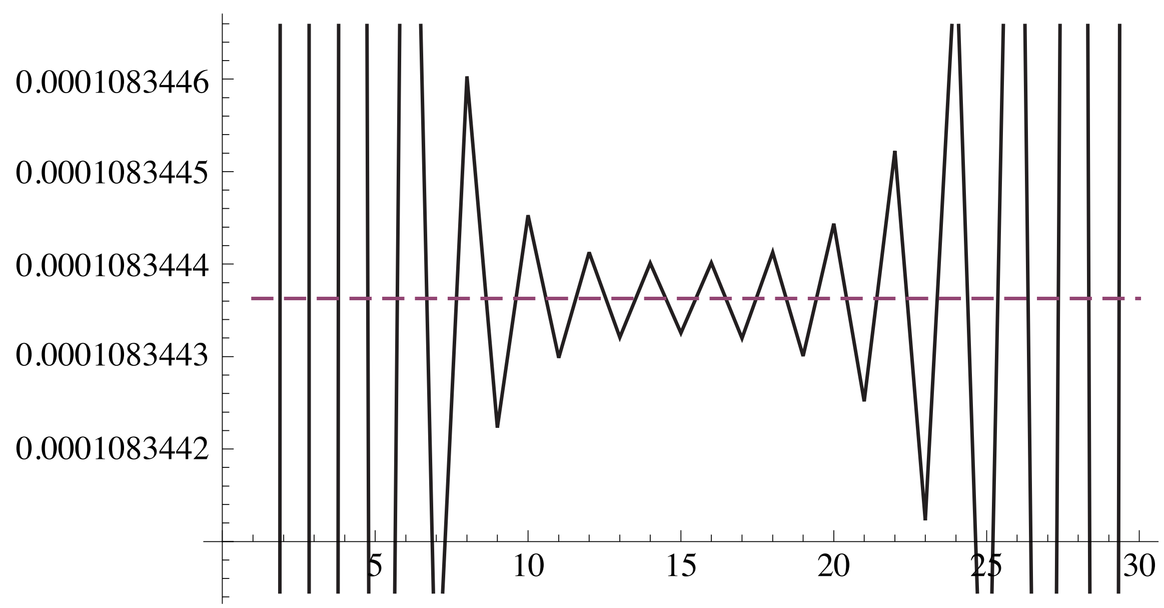
where $\xi = \frac{2}{3} z^{3/2}$ and

$$c_0 = 0, \qquad c_n = \frac{2^n}{3^{3n}} \frac{\Gamma(3n+1/2)}{\Gamma(2n+1)\Gamma(1/2)} \quad (c \geq 1)$$

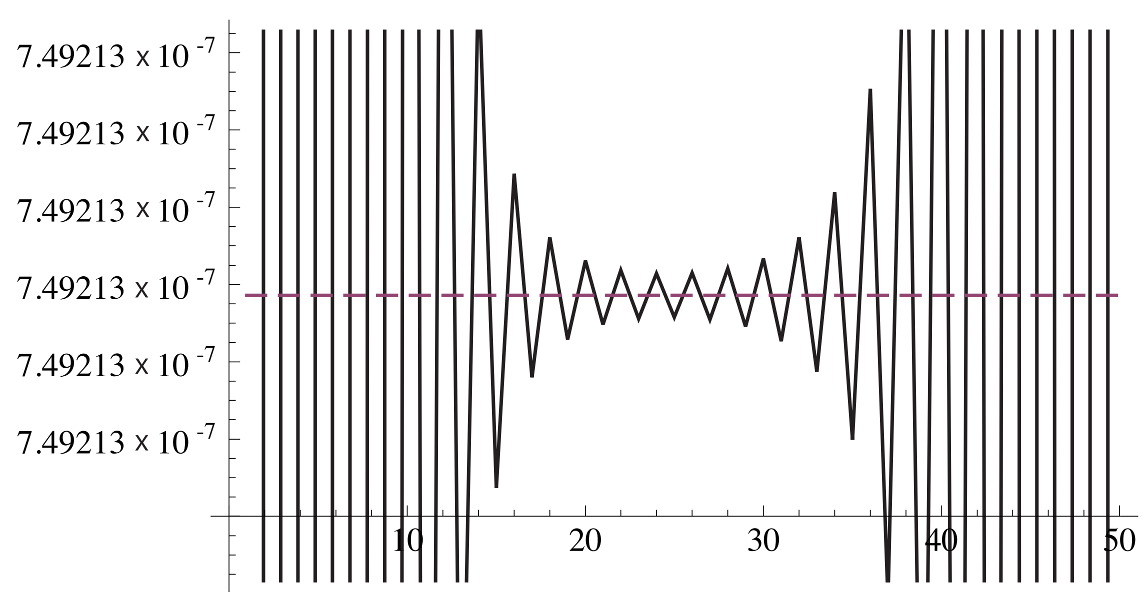



0.006593
0.006592
0.006591
0.006590
2
4
6
8
10
12
14

$z = 3$


0.0001083446
0.0001083445
0.0001083444
0.0001083443
0.0001083442
5
10
15
20
25
30

$z = 5$


7.49213 x 10 -7
7.49213 x 10 -7
7.49213 x 10 -7
7.49213 x 10 -7
7.49213 x 10 -7
7.49213 x 10 -7
10
20
30
40
50

$z = 7$

Asymptotic expansion for $f(z)$

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \quad (z \to \infty \text{ in } \arg z < \pi/2)$$

Since for any n, $z^n e^z = 0$ as $z \to \infty$

$$e^z \sim 0 + \frac{0}{z} + \frac{0}{z^2} + \cdots \quad (z \to \infty \text{ in } \arg z < \pi/2)$$

which implies $g(z) = f(z) + \text{const} \cdot e^z$ has the same asymptotic expansion as $f(z)$.

Or equivalently, asymptotic expansions cannot resolve exponentially small quantities.

" When the cat's away the mice may play. You are the cat and I am the mouse. I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it. I took it up again a few days ago, and after a two or three days' fight, the last of which I sat up till 3, I at last mastered it. I don't say you won't let me work at such things, but you will keep me to more regular hours. A little out of the way now and then does not signify, but there should not be too much of it. It is not the mere sitting up but the hard thinking combined with it..."

G.G. Stokes, *On the discontinuities of arbitrary constants that appear as multipliers of semi-convergent series,* Acta. Math., 26 (1902) 398-397, reprinted in Mathematical and Physical Papers by the later Sir George Gabriel Stokes, Cambridge University Press, 1905 Vol. V, p.283-287

Recent development of asymptotic analysis based on the Borel resummation

- **Resurgent analysis**

  Ecalle, Voros, Balian, Pham-Delabaere, Sauzin, $\cdots$

- **Exponential asymptotics**

  Dingle, Berry-Howls, King, Daalhuis, Olver, $\cdots$

- **Exact WKB analysis**

  Sato-Aoki-Kawai-Koike-Takei, $\cdots$

Resurgent theory : a sketch

divergent seres → Borel transform → (analytic continuation) → Laplace transform

singularities on the Borel plane

↕

Stokes phenomenon

Construction of global (WKB) solutions

||

connection formula **(= local)**

+

Riemann sheet structure of the Borel transform **(= global)**

Borel resummation method — an example —

Euler equation

$$-\frac{df(z)}{dz} + f(z) = \frac{1}{z}$$

Formal power series solution

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{z^{n+1}}$$

Laplace transform

$$\mathcal{L}\zeta^n \equiv \int_0^{\infty} \zeta^n e^{-z\zeta} d\zeta = \frac{n!}{z^{n+1}} \quad \Rightarrow \quad \mathcal{L}^{-1}\frac{n!}{z^{n+1}} = \zeta^n$$

Borel transform (=inverse Laplace transform) of $f(x)$

$$f_B(\zeta) \equiv \mathcal{L}^{-1} f(z) = \mathcal{L}^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{z^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \zeta^n = \frac{1}{1+\zeta}$$

Borel sum of $f(z)$

$$f(z) = \mathcal{L} f_B(\zeta) = \int_0^{\infty} f_B(\zeta) e^{-z\zeta} d\zeta = \int_0^{\infty} \frac{e^{-z\zeta}}{1+\zeta} d\zeta$$

Analytical continuation and Stokes phenomenon

Analytical continuation of Borel sum of $f(z)$

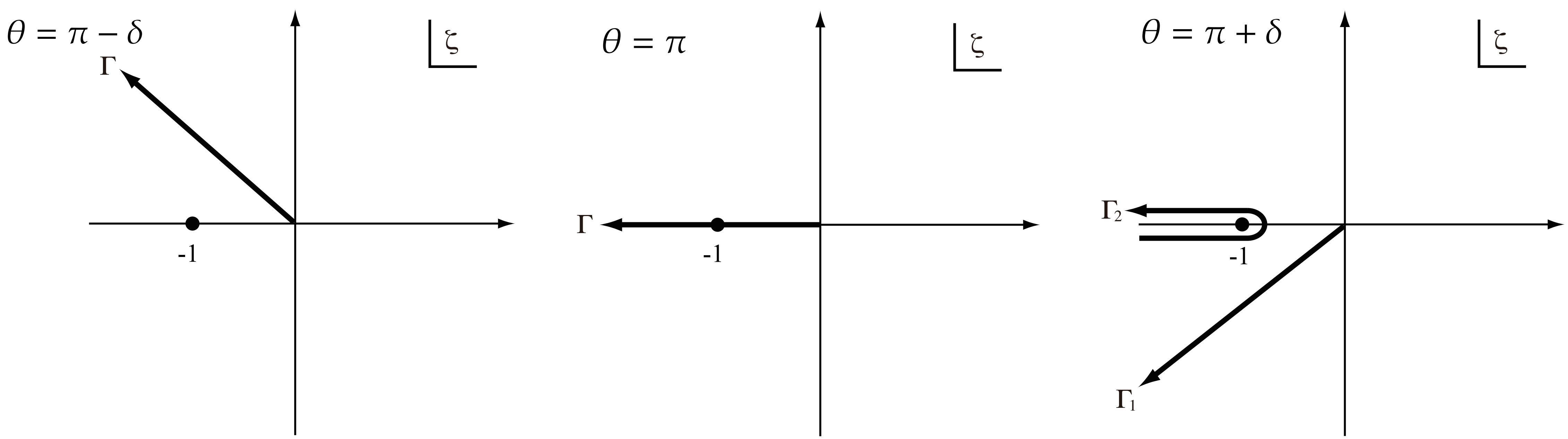
$$f(z) = \int_0^{\infty e^{i\theta}} \exp(-z\zeta) f_B(\zeta) d\zeta$$

For $\theta = \pi - \delta$

$$f(z) = \int_\Gamma \frac{e^{-z\zeta}}{1+\zeta} d\zeta$$

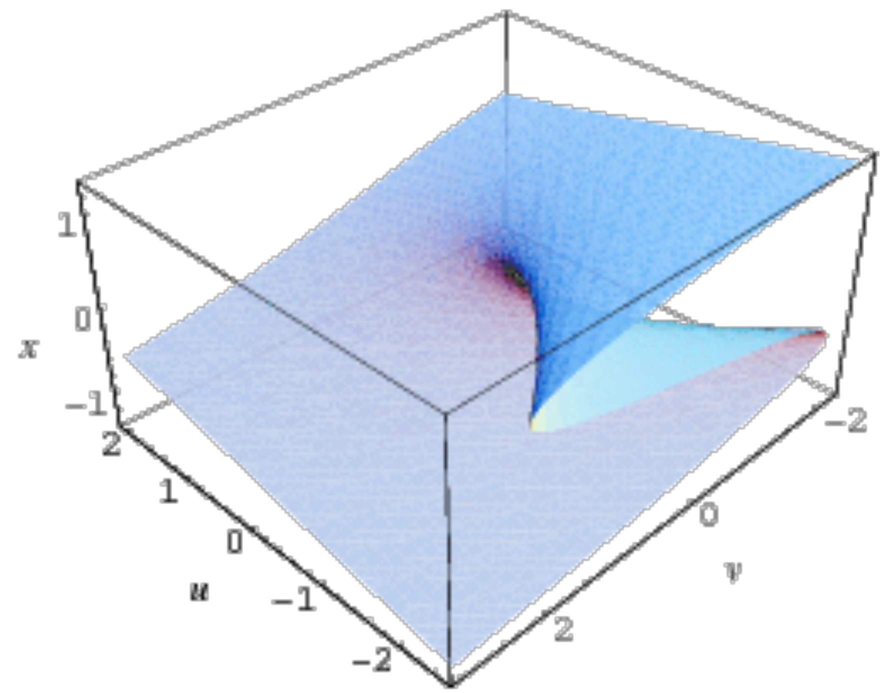
For $\theta = \pi + \delta$

$$f(z) = \underbrace{\int_{\Gamma_1} \frac{e^{-z\zeta}}{1+\zeta} d\zeta}_{\int_0^{\infty e^{i\theta}} \frac{e^{-z\zeta}}{1+\zeta} d\zeta} + \underbrace{\int_{\Gamma_2} \frac{e^{-z\zeta}}{1+\zeta} d\zeta}_{2\pi i e^z}$$

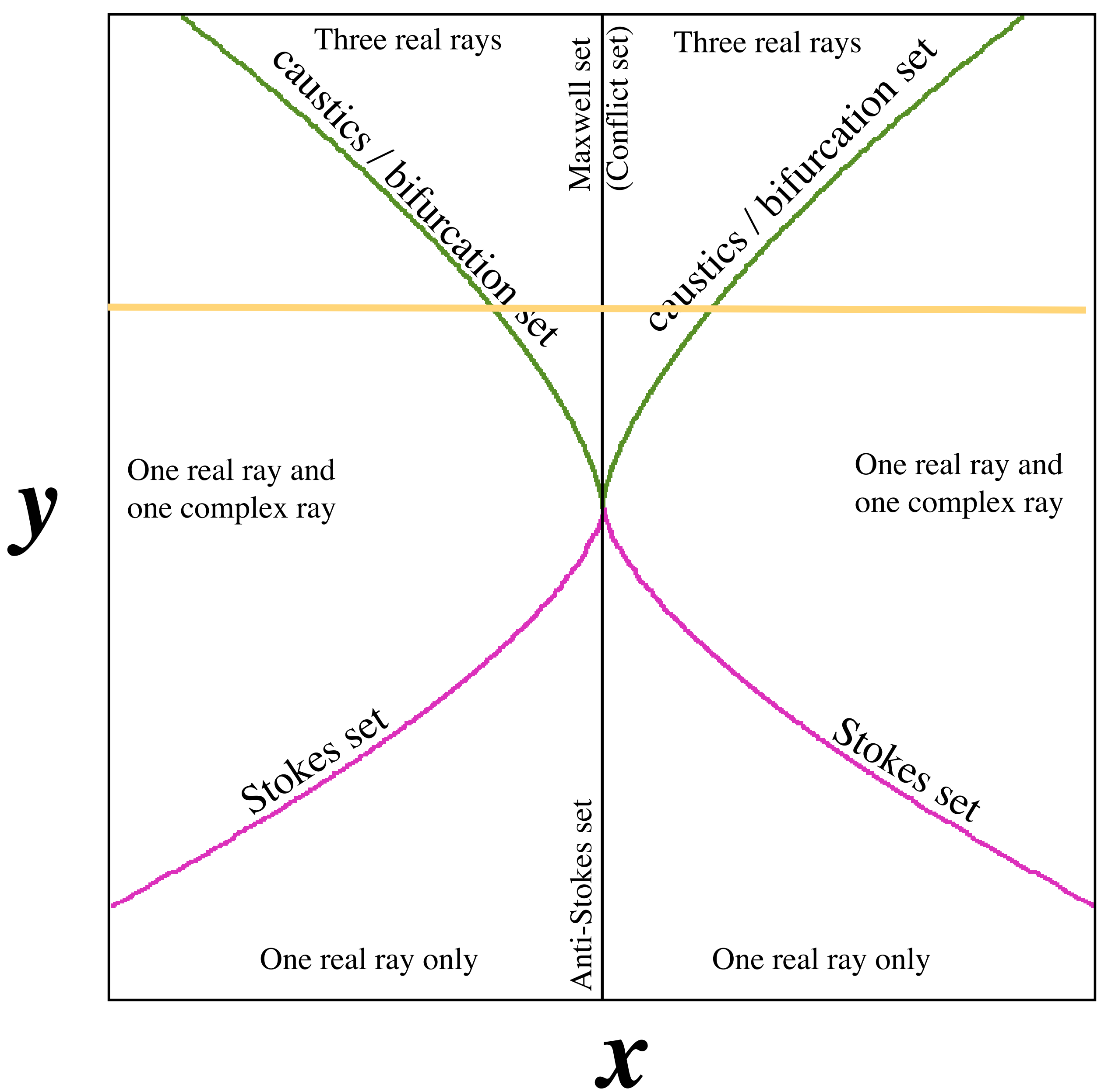


Much less trivial example — Cusp —

$$S(\xi; x, y) = \frac{1}{4}\xi^4 + \frac{1}{2}y\xi^2 + x\xi$$



"The Stokes set of the cusp diffraction catastrphe"
FJ. Wright J.Phys.A 13 (1980) 2913



Stokes set

$$\mathrm{Re}\, S(\xi_1(X)) = \mathrm{Re}\, S(\xi_2(X))$$

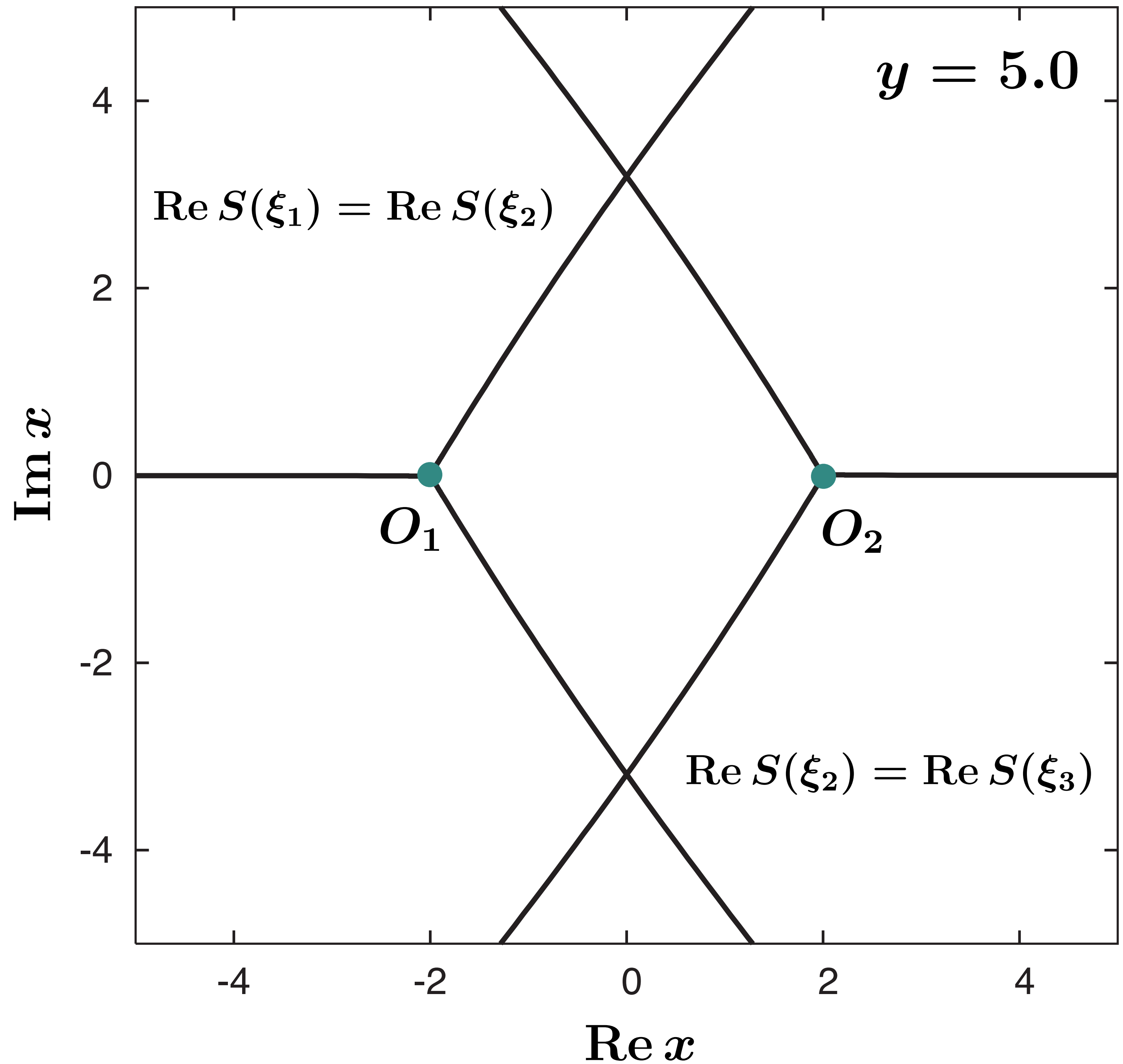
Bifurcation set (caustics)

$$\frac{\partial S(\xi, X)}{\partial \xi} = \frac{\partial^2 S(\xi, X)}{\partial \xi^2} = 0$$

For each turning point O_i ($i = 1, 2$), find the Stokes lines emanating from it, using the condition

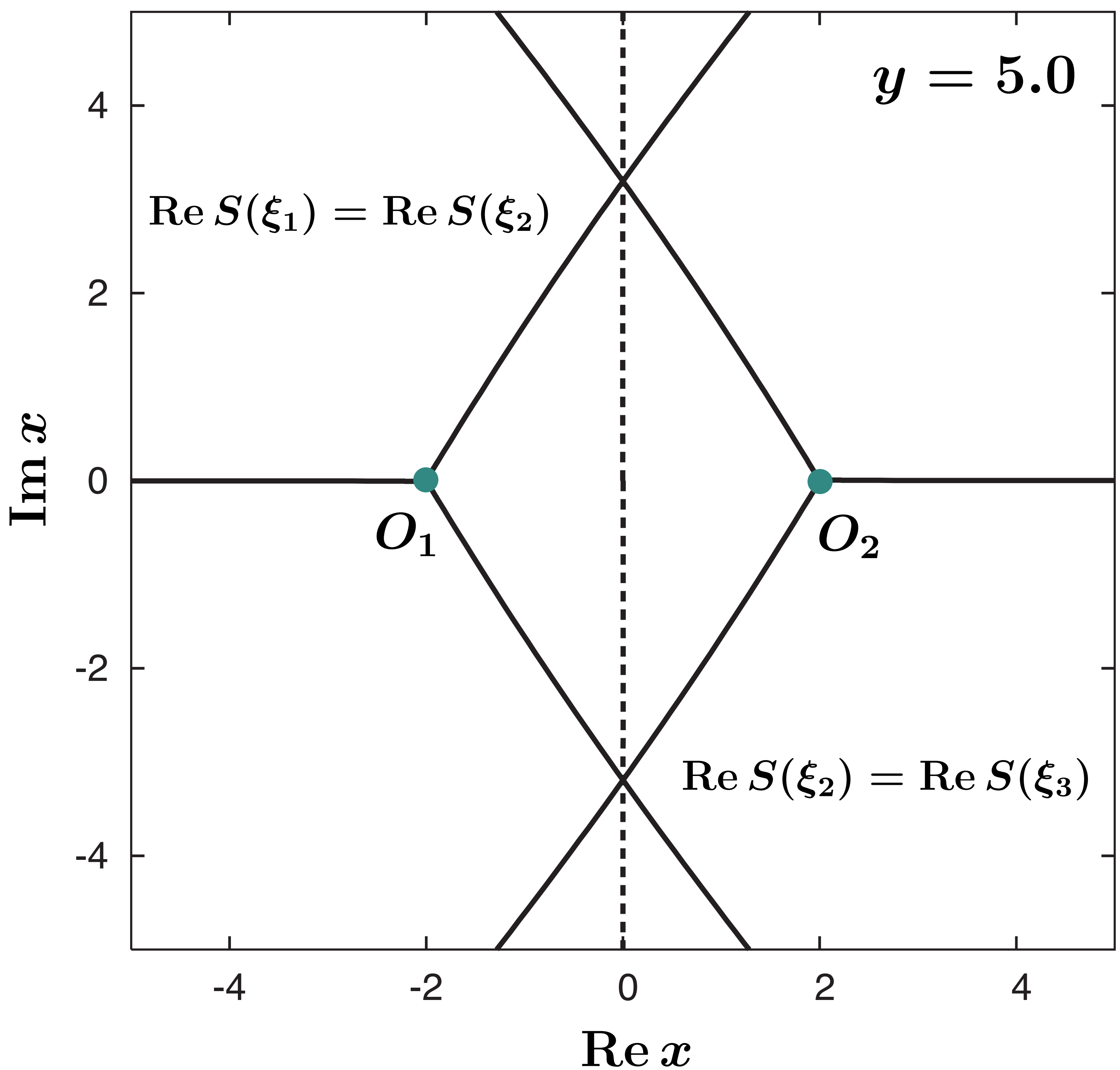
$$\mathrm{Re}\, S(\xi_\alpha(X)) = \mathrm{Re}\, S(\xi_\beta(X)) \qquad (\alpha \neq \beta)$$

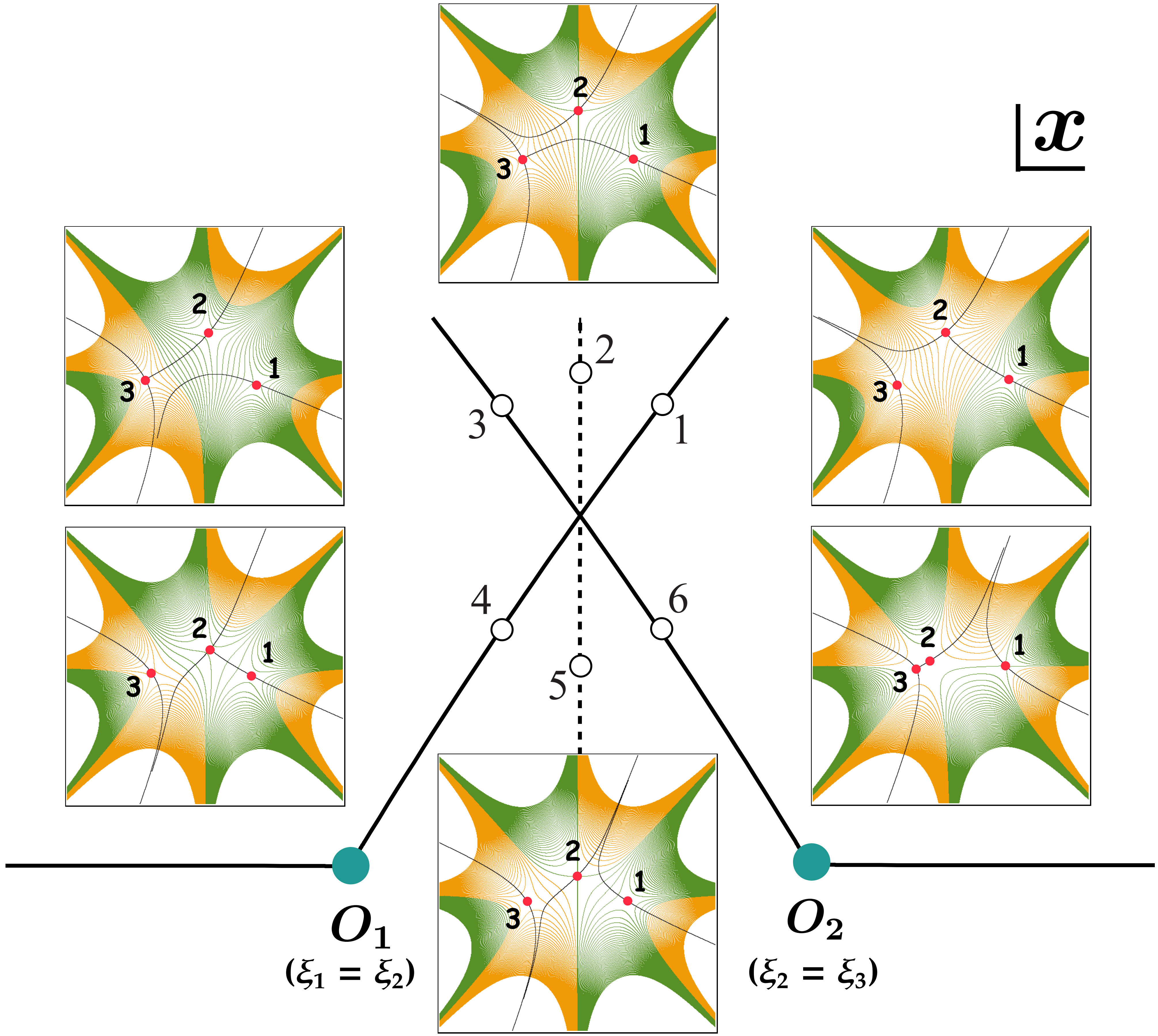
Note, for example, O_1 is the turning point associated with ξ_1 and ξ_2
O_2 associated with ξ_2 and ξ_3 **, but why not between ξ_1 and ξ_3?**



The new Stokes curve, not noticed in the theory of asymptotic expansions until *Berk-Nevins-Roberts*(1982) and *Aoki-Kawai-Takei*(1994)

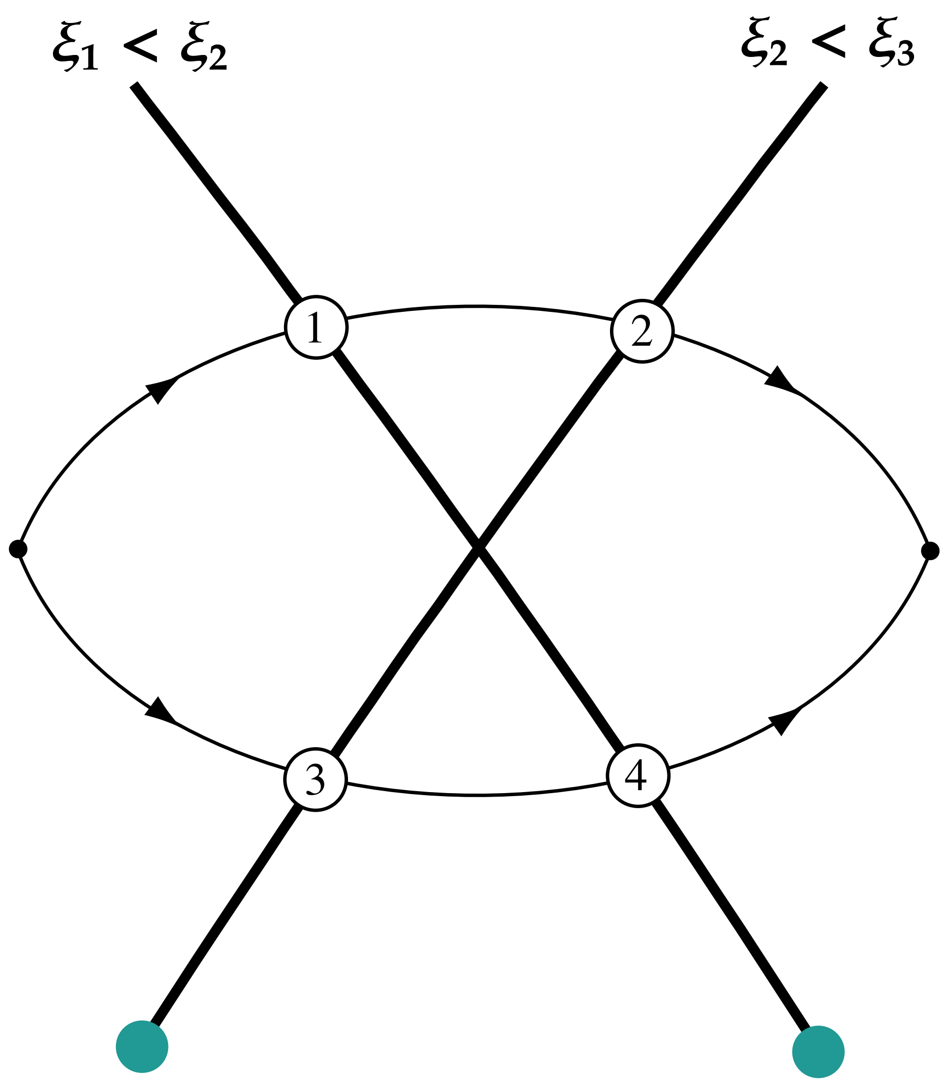
$\mathrm{Re}\, S(\xi_1) = \mathrm{Re}\, S(\xi_3)$





x
1
2
3
4
5
6
O_1
$(\xi_1 = \xi_2)$
O_2
$(\xi_2 = \xi_3)$

Why does the Stokes connection occur on one side, but not on the other side ?



$$I(k,X) = \sum_i \int_{C_i} d\xi \exp\left[ikS(\xi,X)\right]$$
$$= \sum_i I_i(k,X)$$

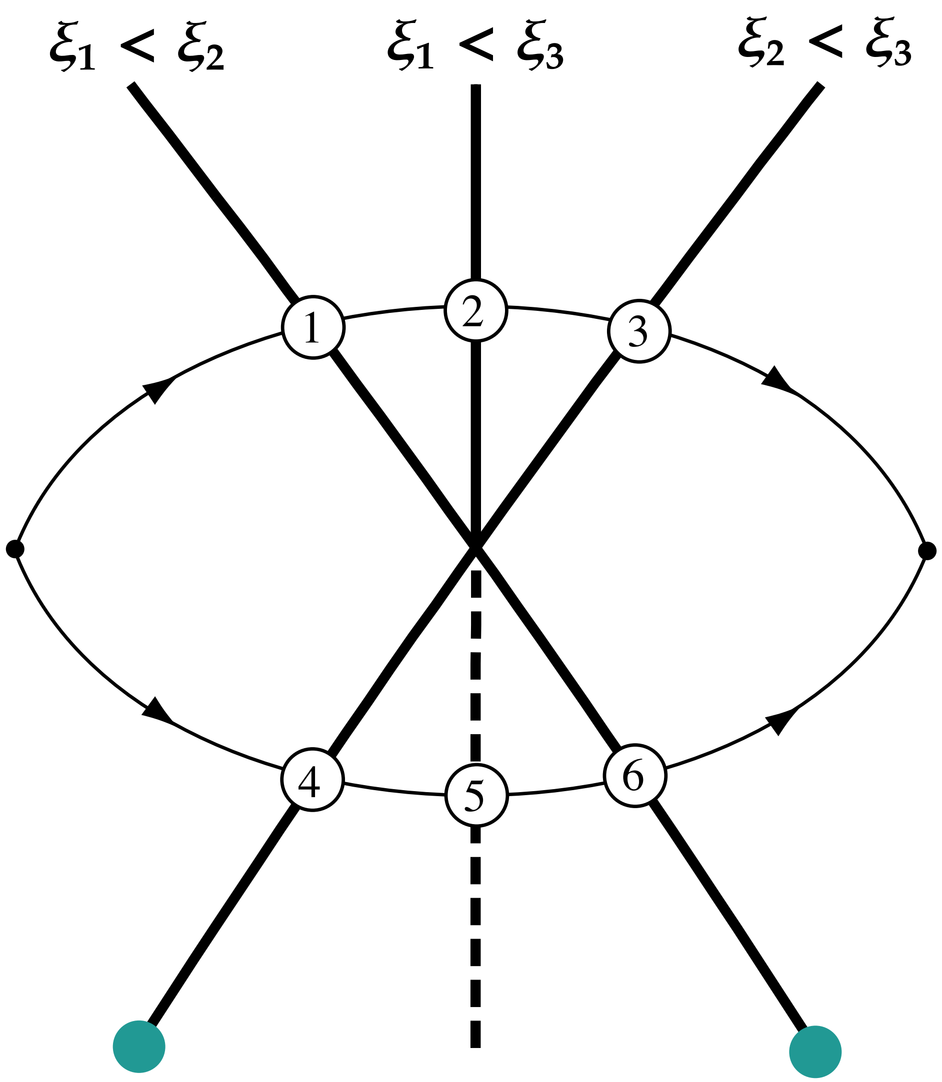
C_i : steepest descent line passing through the saddle ξ_i

$$I_3 \xrightarrow{①} I_3 \xrightarrow{②} I_3 + \sigma_2 I_2$$

$$I_3 \xrightarrow{③} I_3 + \sigma_2 I_2 \xrightarrow{④} I_3 + \sigma_2(I_2 + \sigma_1 I_1)$$

Univaluedness not satisfied ...

Why does the Stokes connection occur on one side, but not on the other side ?



$$I(k, X) = \sum_i \int_{C_i} d\xi \exp\left[ikS(\xi, X)\right] = \sum_i I_i(k, X)$$

C_i : steepest descent line passing through the saddle ξ_i

$$I_3 \xrightarrow{1} I_3 \xrightarrow{2} I_3 + \sigma_1' I_1 \xrightarrow{3} I_3 + \sigma_2 I_2 + \sigma_1' I_1$$

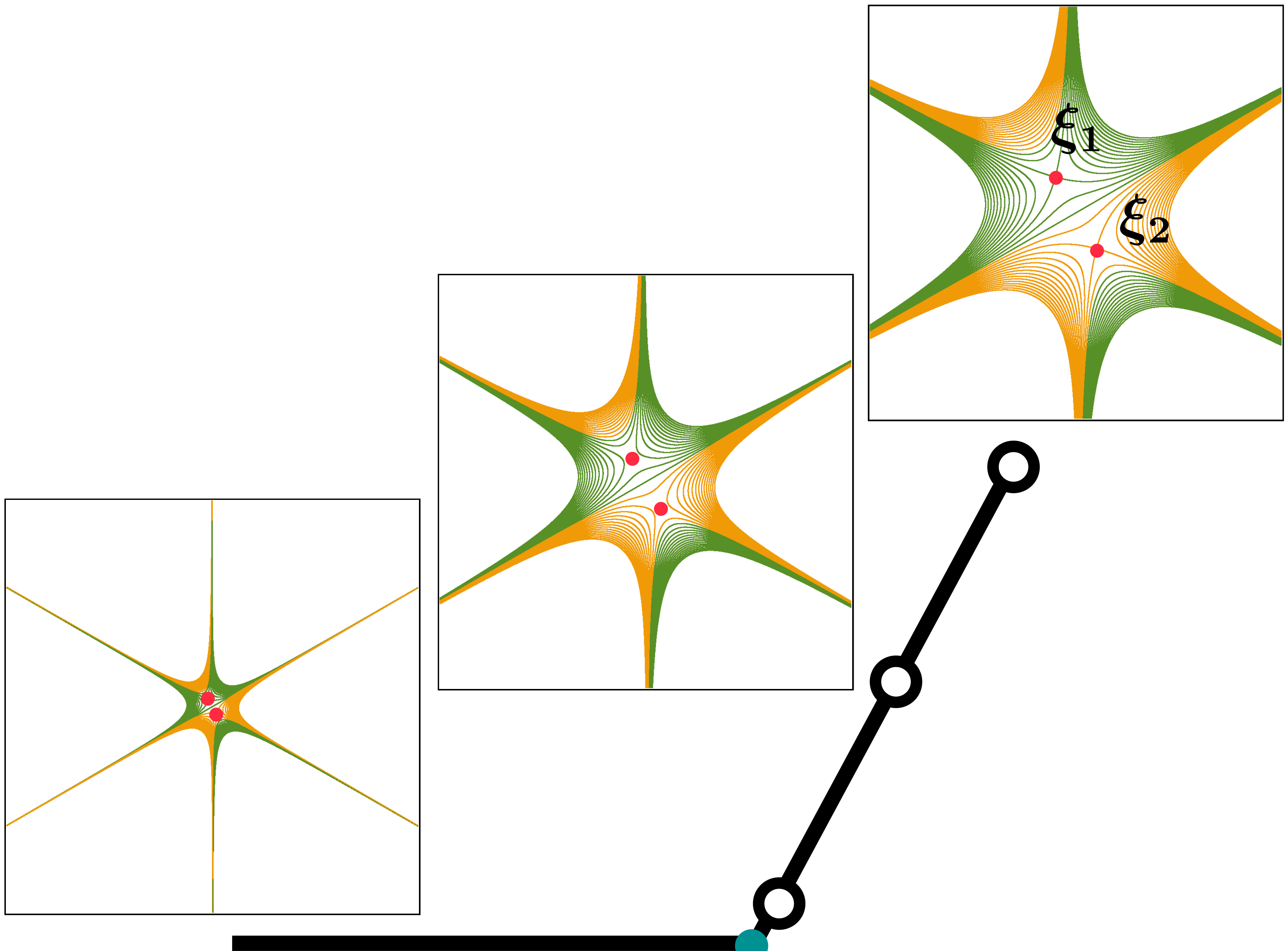
$$I_3 \xrightarrow{4} I_3 + \sigma_2 I_2 \xrightarrow{5} I_3 + \sigma_2 I_2 \xrightarrow{5} I_3 + \sigma_2 (I_2 + \sigma_1 I_1)$$

Univaluedness recovered if

$$\sigma_1 \sigma_2 = \sigma_1'$$

From where does the new Stokes curve emanate ?

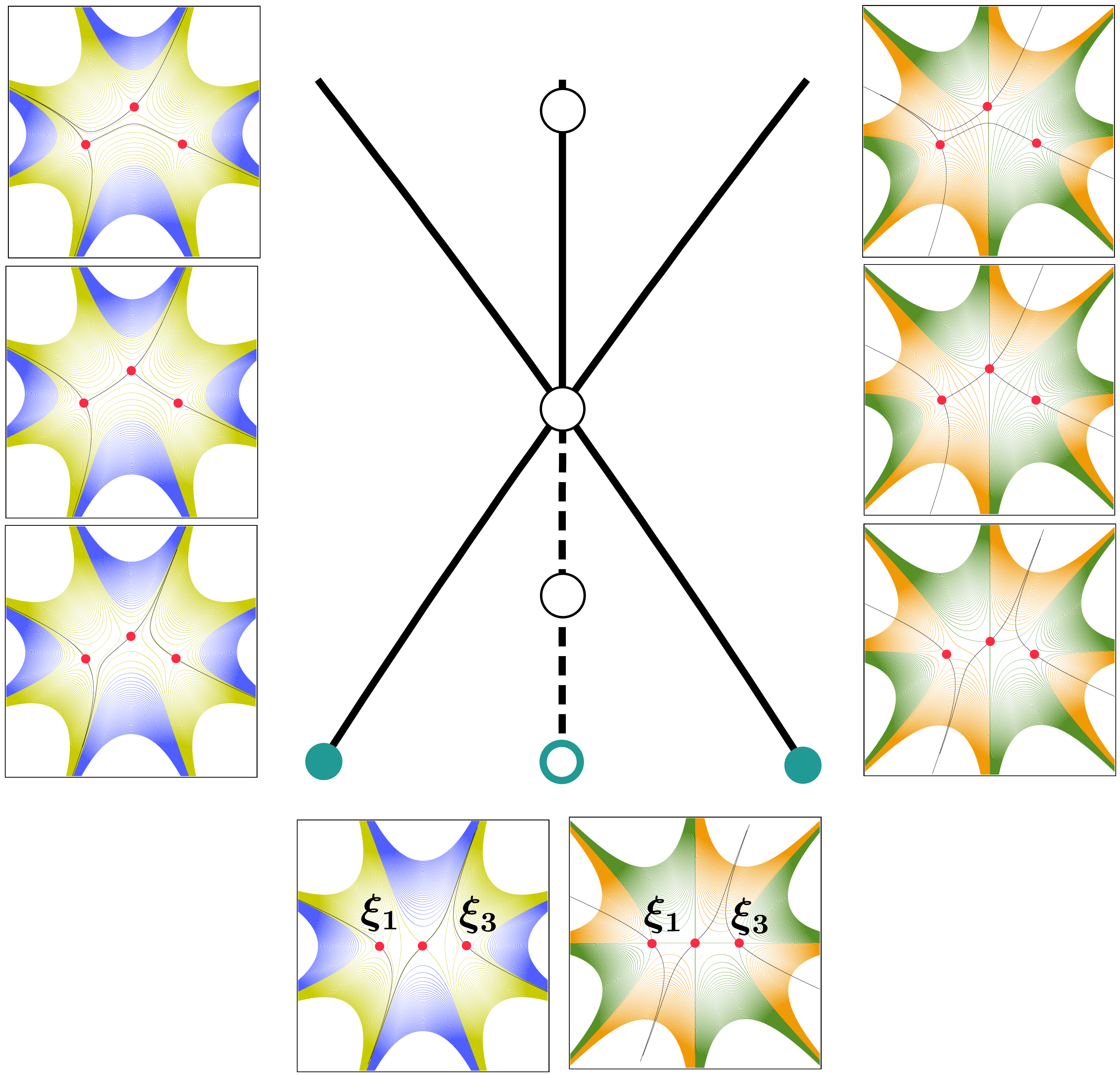
—   Ordinary turning point case   —



Two saddles collide at the turning point

$\mathrm{Re}\, S(\xi_1) = \mathrm{Re}\, S(\xi_2)$ $\mathrm{Im}\, S(\xi_1) = \mathrm{Im}\, S(\xi_2)$

From where does the new Stokes curve emanate ?



$\mathrm{Re}\, S(\xi_1) = \mathrm{Re}\, S(\xi_3) \quad \mathrm{Im}\, S(\xi_1) = \mathrm{Im}\, S(\xi_3)$

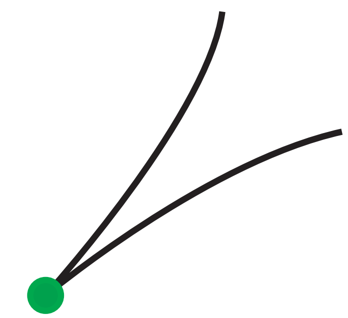
Virtual turning point (Aoki-Kawai-Takei 1992)

The virtual turning point as the self-intersection point of bicharacteristic curves

Definition (Aoki-Kawai-Takei) The *virtual turning point* is a self-intersection point of the projection of a bicharacteristic strip (= solution curves of the bicharacteristic equation) onto (x, y) plane.

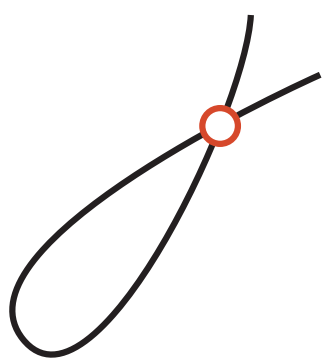
- ordinary turning points:

$$x'(t) = y'(t) = 0 \text{ at } t = t_0$$



- virtual turning points:

$$x(t) = x(s) \text{ and } y(s) = y(t) \text{ at } t \neq s$$



Higher-order differential equations and its Borel transform

m-th order differential eq.

$$H\psi = 0 \quad \text{where} \quad H = \sum_{0\le i\le m, 0\le j\le n} a_{ij} x^j k^{m-i} \left(\frac{d}{dx}\right)^i$$

Its Borel transform

$$H_B \psi_B = 0 \quad \text{where} \quad H_B = \sum_{0\le i\le m, 0\le j\le n} a_{ij} \frac{\partial^{m-j}}{\partial y^{m-j}} \left(-\frac{\partial}{\partial \xi}\right)^j \xi^i$$

where ψ_B is the Borel transform of ψ.

Let P_B the principal symbol for the operator H_B. Then the bicharacteristic equations for H_B (= Hamilton-Jacobi equation) are written as

$$\frac{dx}{dt} = \frac{\partial P_B}{\partial \xi}, \quad \frac{d\xi}{dt} = -\frac{\partial P_B}{\partial x}, \quad \frac{dy}{dt} = \frac{\partial P_B}{\partial k}, \quad \frac{dk}{dt} = -\frac{\partial P_B}{\partial y}, \quad P_B(x, \xi, \eta) = 0$$

A recipe for finding a complete Stokes geometry

step1 Find all the ordinary turning points.

step2 Draw the ordinary Stokes lines emanating from them.

step3 Find intersection points between ordinary Stokes lines.

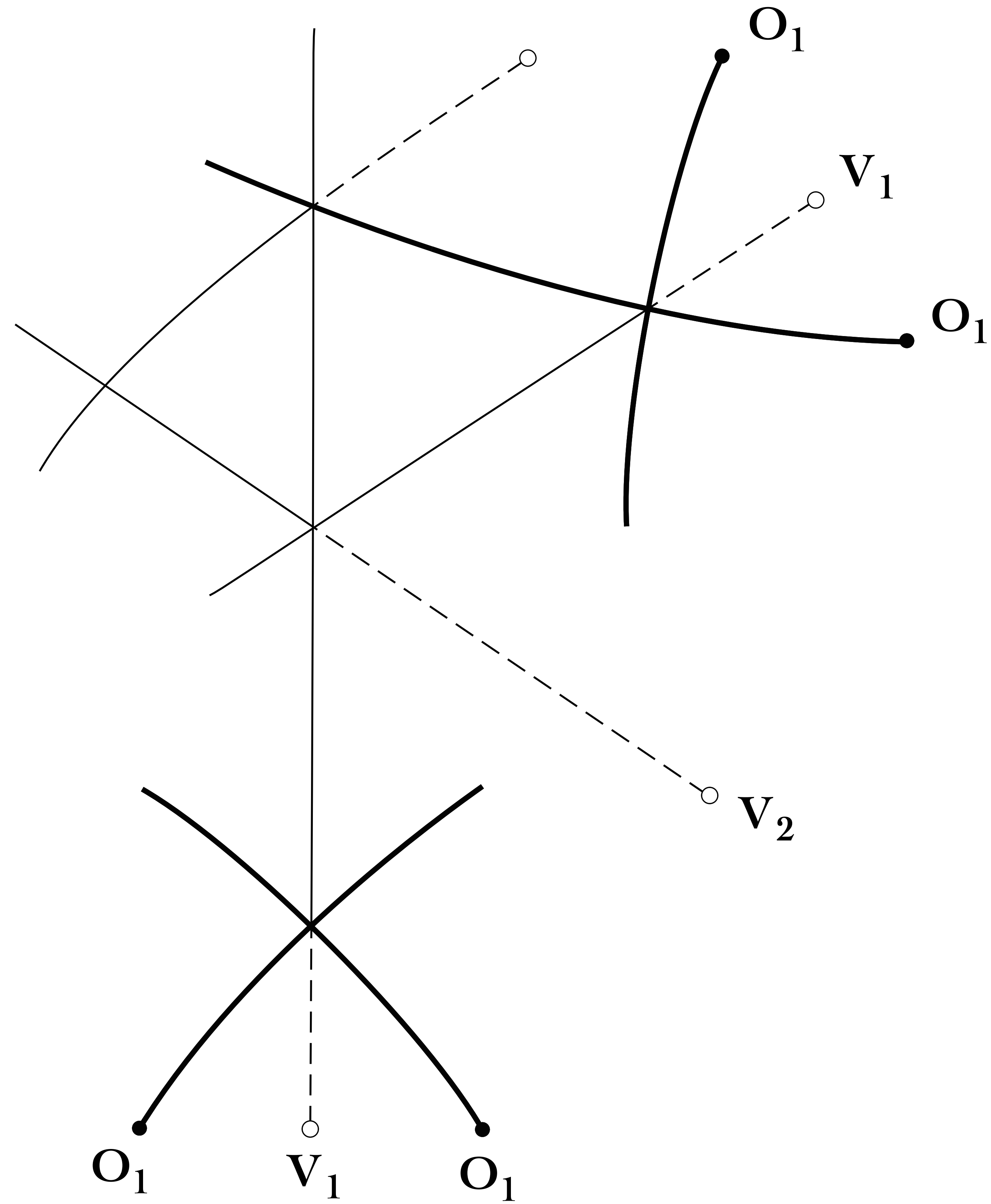
step4 For each intersection point, find a virtual turning point.

step5 Draw new Stokes lines emanating each virtual turning point.

step6 Determine the nature of connection around each crossing point by checking the univaluedness condition.

step7 Find intersection points between Stokes lines emanating from 1-st order and those from 0-th order turning points.

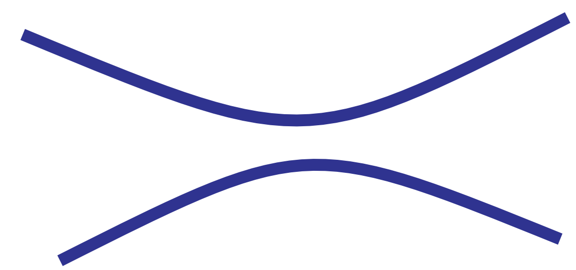
step8 Repeat this procedure until relevant Stokes lines do not appear.



Model for nonadiabatic transition

Time-dependent 2-level problem (Landau-Zener)

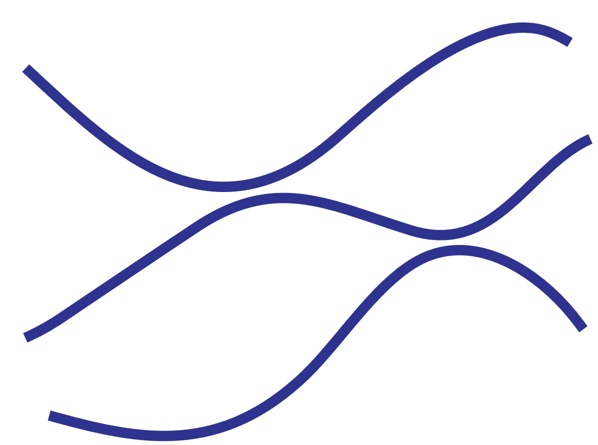
$$i\hbar \frac{d}{dt}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \rho_1(t) & c_{12} \\ \overline{c_{12}} & \rho_2(t) \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$



$$\rho_1(t) = b_1 t + a, \quad \rho_2(t) = b_2 t, \quad c_{12} = \text{const.}$$

Time-dependent multi-level problem (Aoki-Kawai-Takei)

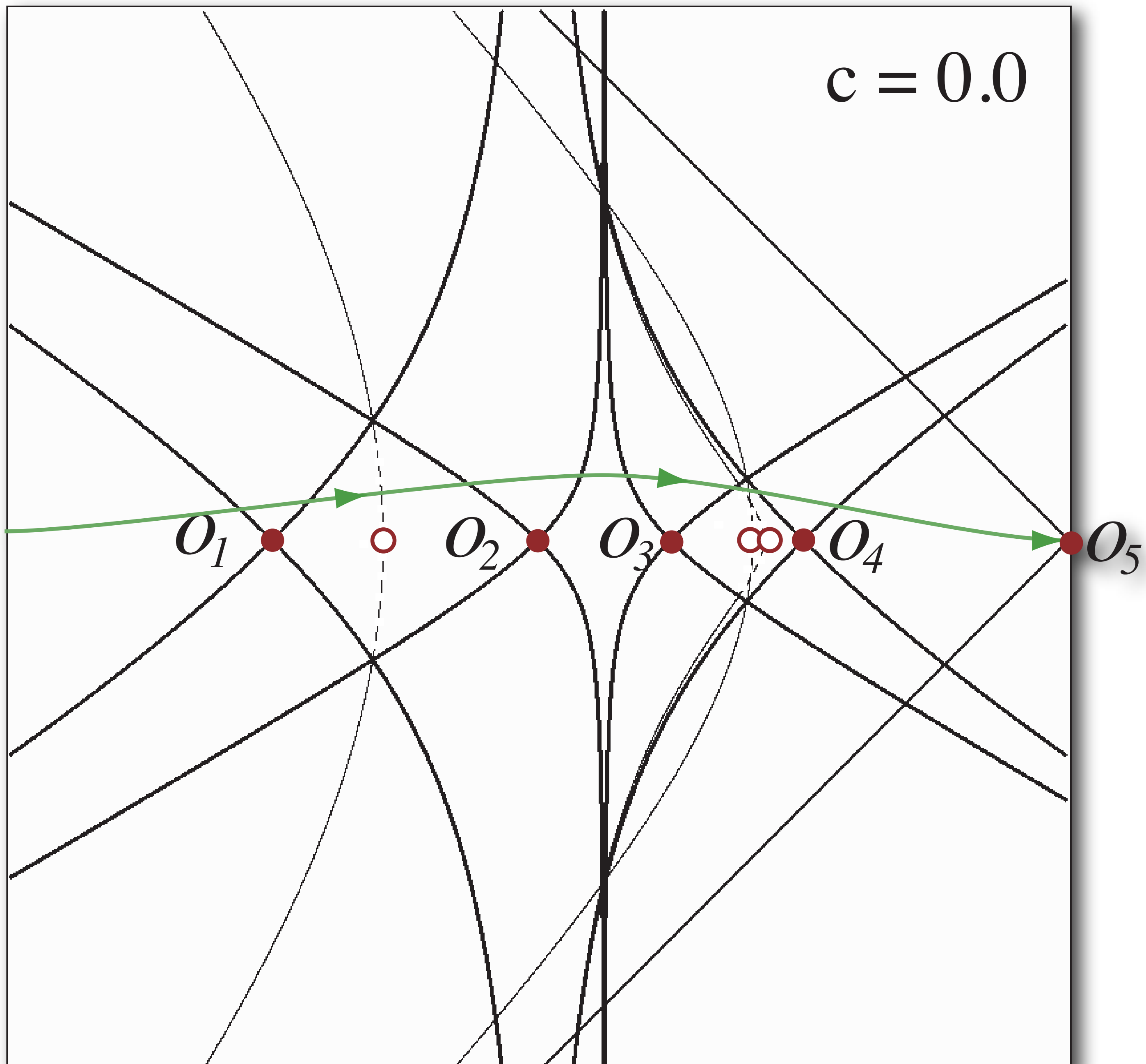
$$i\hbar \frac{d}{dt}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \rho_1(t) & c_{12} & c_{13} \\ \overline{c_{12}} & \rho_2(t) & c_{23} \\ \overline{c_{31}} & \overline{c_{32}} & \rho_3(t) \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$



i) $\rho_1(t) = b_1 t + a, \quad \rho_2(t) = b_2 t, \quad \rho_3(t) = b_3 t$

ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$

ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$



Possible situations where new Stokes play a role

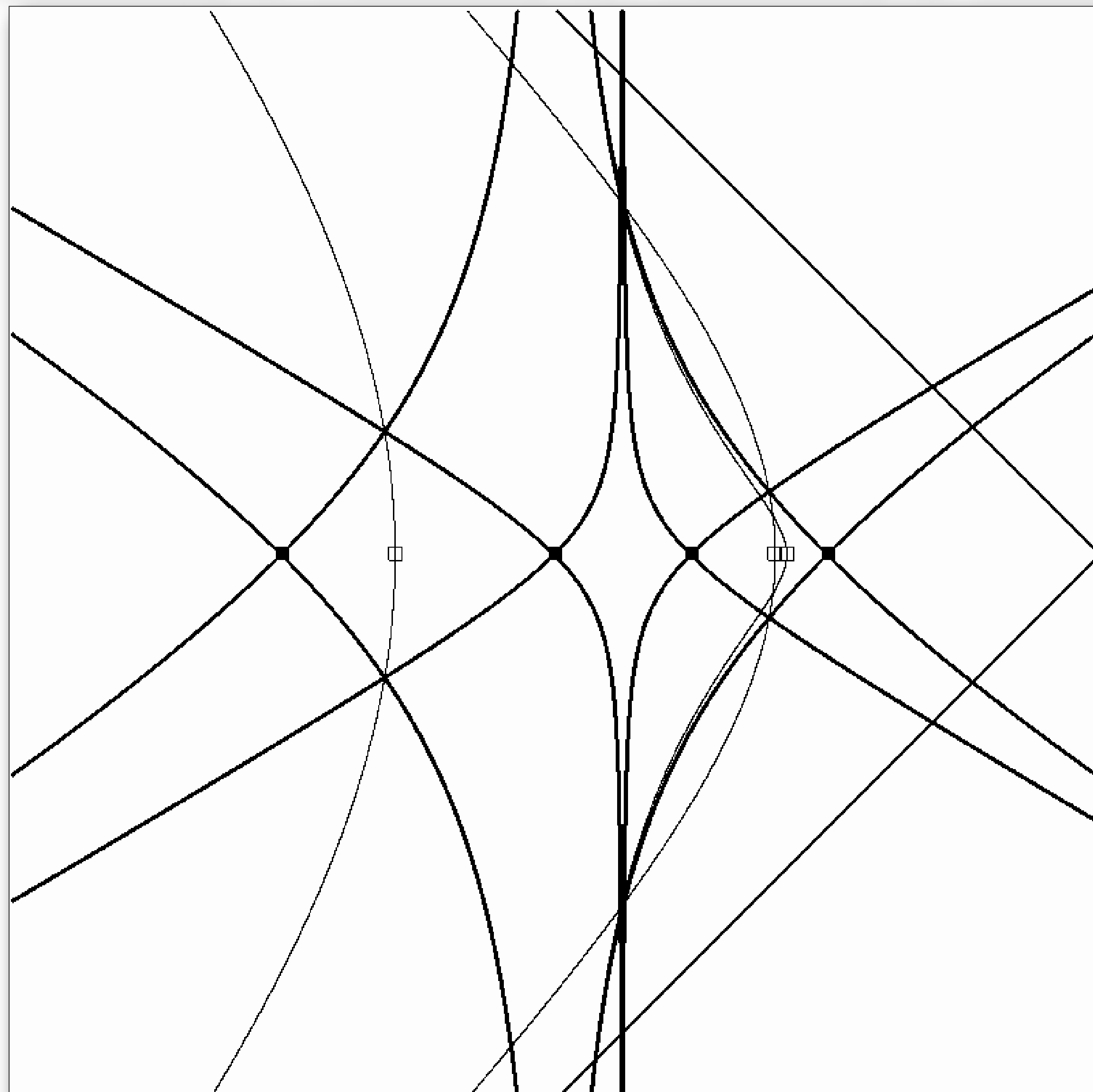
Case 1) We can avoid crossing new Stokes curves, but need information on new Stokes curves and virtual turning points in order to find a connection path which does not pass any new Stokes curves.

Case 2) We cannot avoid crossing new Stokes curves because of the presence of a chain of active new Stokes curves.

ii) $\rho_1(t) = 1, \quad \rho_2(t) = \dfrac{t}{2}, \quad \rho_3(t) = t^2 + c$

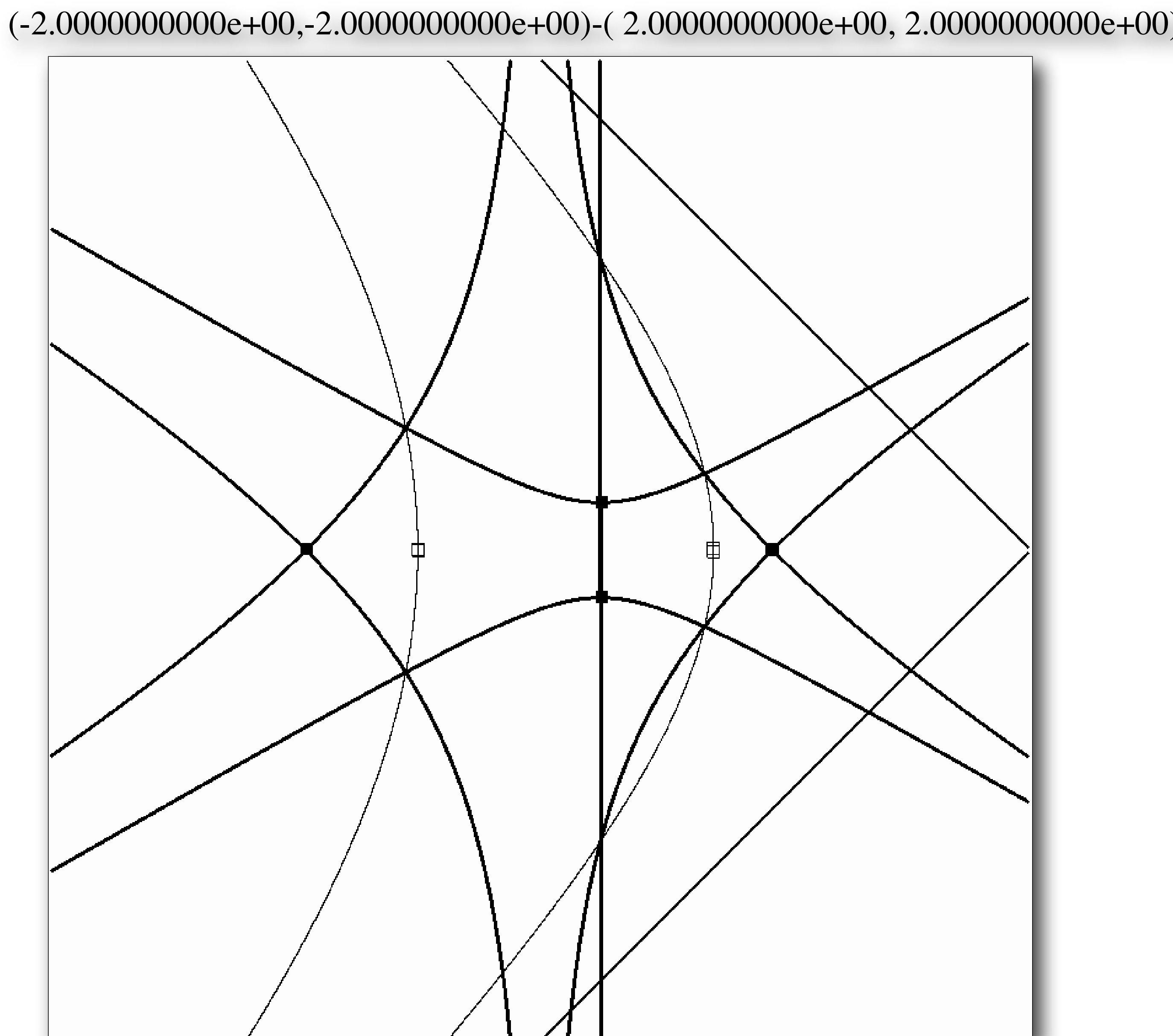
c = 0.0

(-2.0000000000e+00,-2.0000000000e+00)-(2.0000000000e+00, 2.0000000000e+00)



ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$

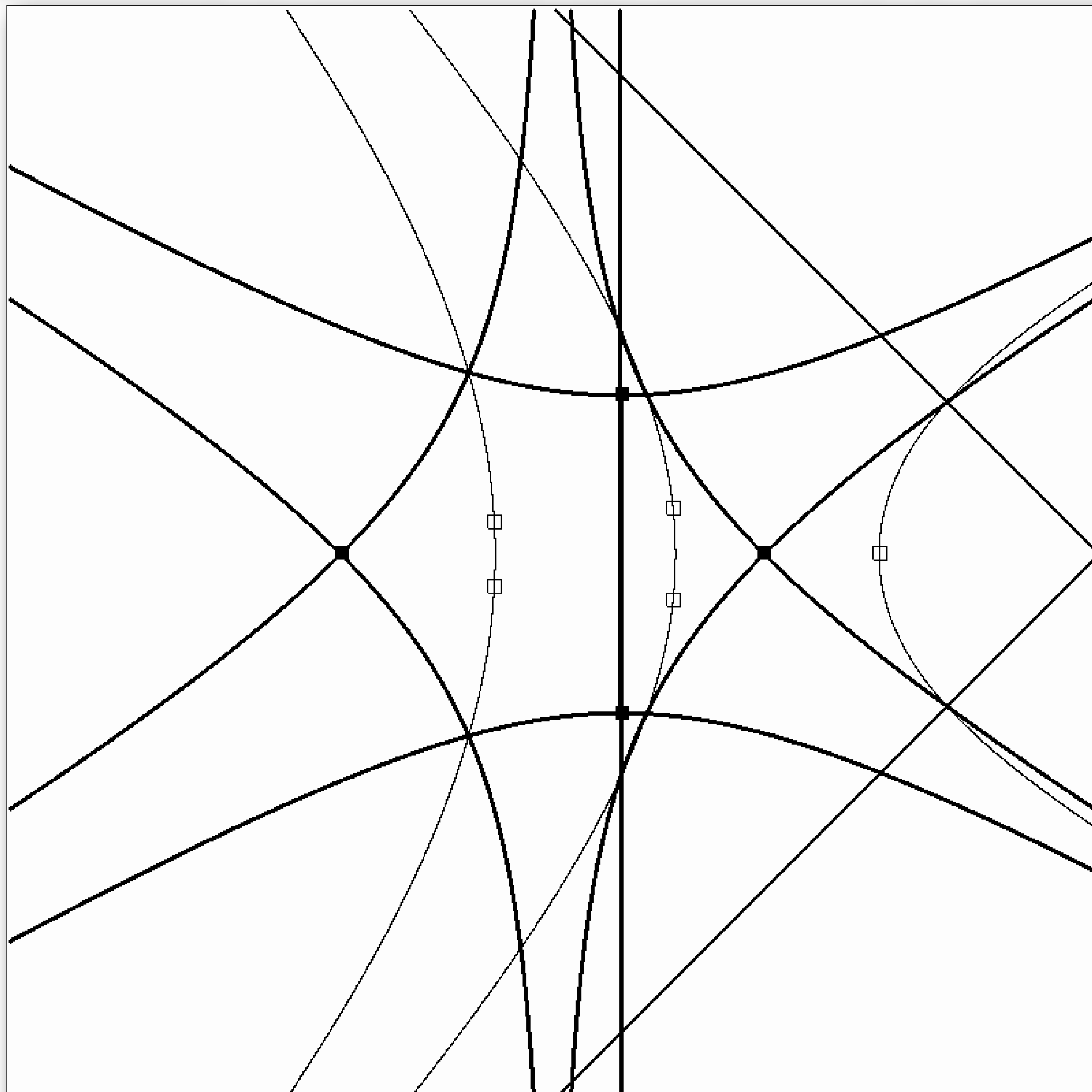
c = 0.1



ii) $\rho_1(t) = 1, \quad \rho_2(t) = \dfrac{t}{2}, \quad \rho_3(t) = t^2 + c$

c = 0.4

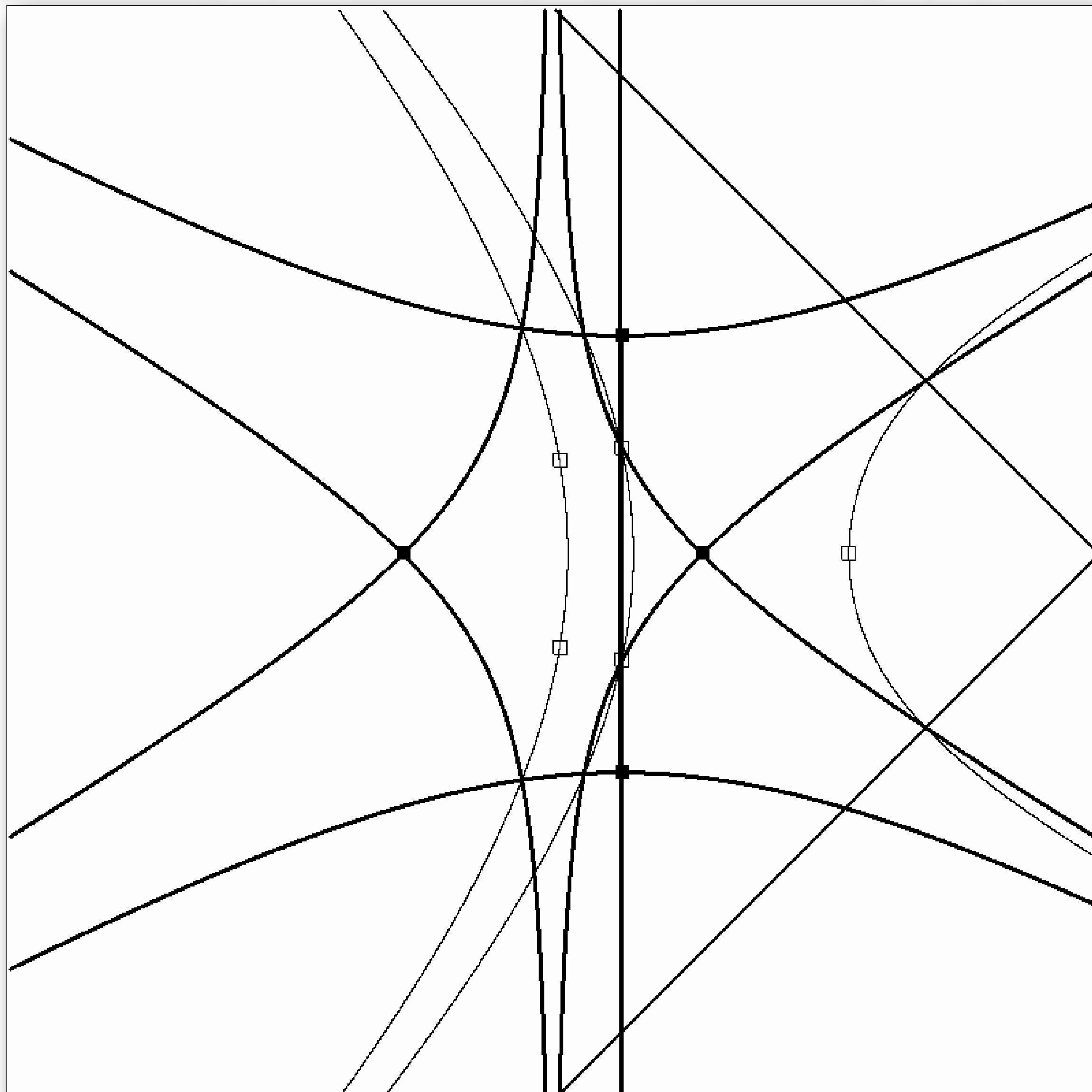
(-2.0000000000e+00,-2.0000000000e+00)-(2.0000000000e+00, 2.0000000000e+00)



ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$

c = 0.7

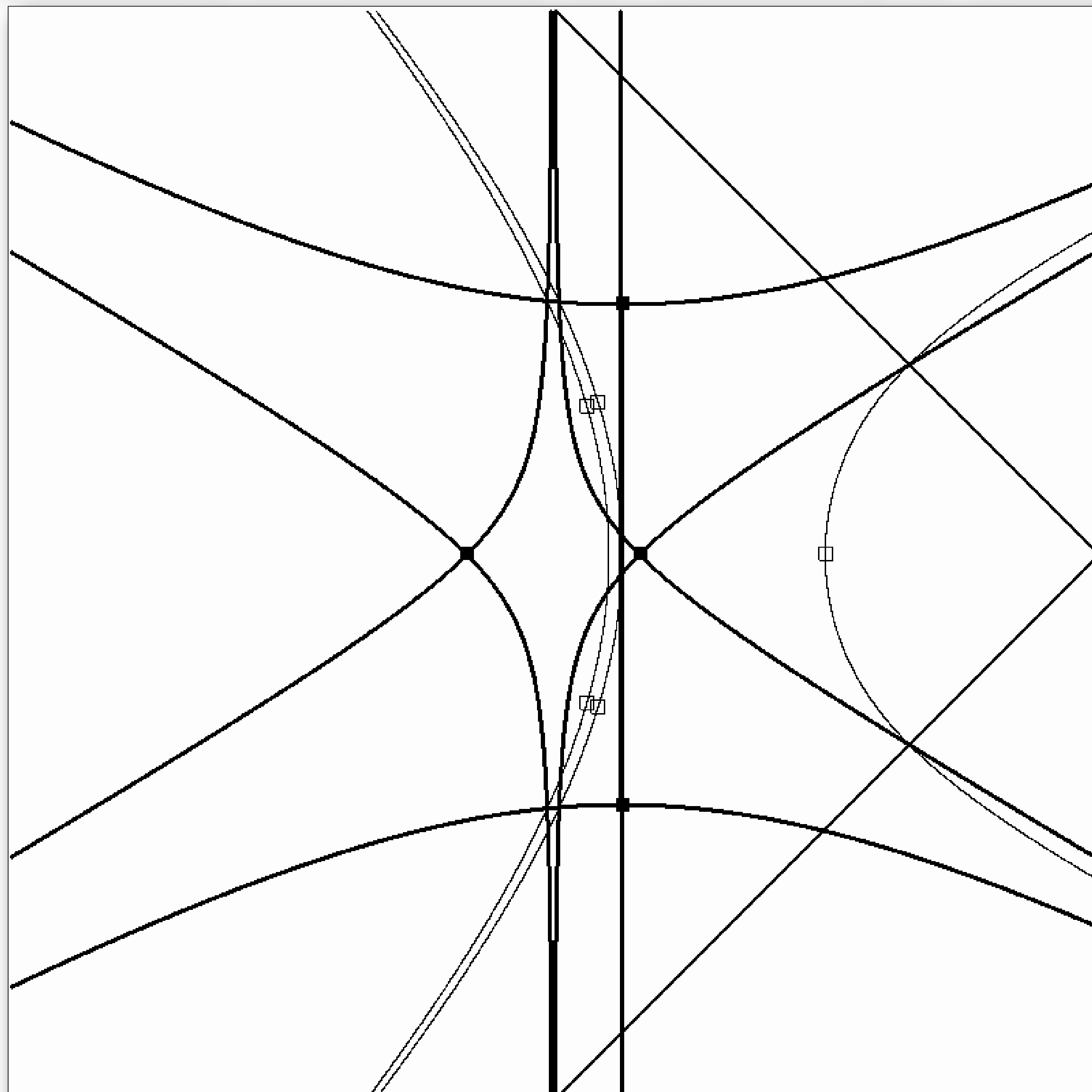
(-2.0000000000e+00,-2.0000000000e+00)-(2.0000000000e+00, 2.0000000000e+00)



ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$

$c = 0.9$

(-2.0000000000e+00,-2.0000000000e+00)-(2.0000000000e+00, 2.0000000000e+00)



ii) $\rho_1(t) = 1, \quad \rho_2(t) = \frac{t}{2}, \quad \rho_3(t) = t^2 + c$

c = 1.2

(-2.0000000000e+00,-2.0000000000e+00)-(2.0000000000e+00, 2.0000000000e+00)