

From Monte Carlo simulation to volatility filtering: The evolution of simulation methods in market risk

XXXXI Heidelberg Physics Graduate Days

Heidelberg, 10th October 2018

Agenda

» What is market risk and how can we measure it ?	2
› Examples and definitions	3
› Measures for market risk	6
› Backtesting – assessing the quality of a market risk model	9
» Current market standards	13
› General setup for simulations	14
› Historical simulation and Monte Carlo simulation	17
» Advanced simulation methods	22
› Advanced variants of historical simulation	24
› Improving Monte Carlo	35
» Summary and outlook	52

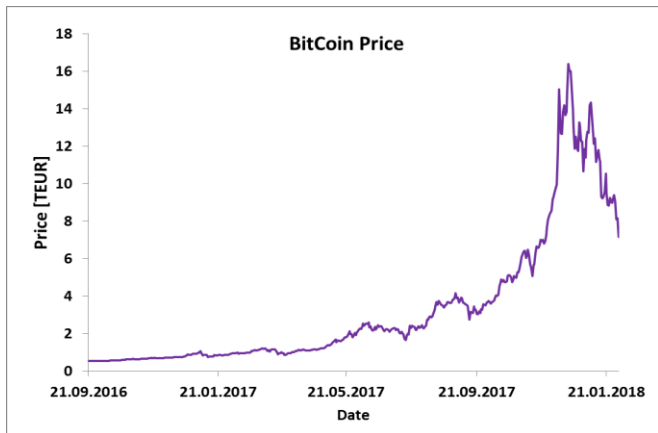
What is market risk and how can we measure it ?

Examples and definitions

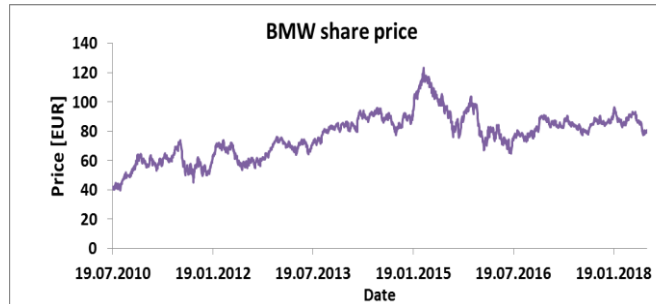
What is market risk ?

- » Market risk is hidden behind many everyday phenomena and news but becomes 'visible' every now and then

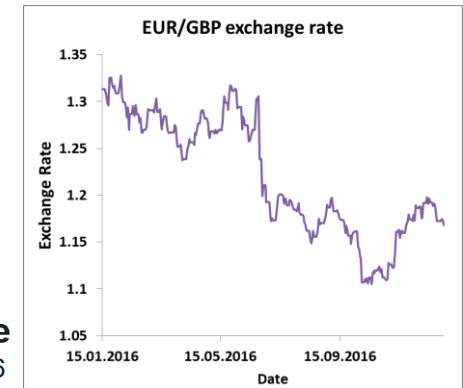
Bitcoin Prices Extend Losses, Hit New 2018 Low
Forbes, 23/06/2018



Trump's Tariff Tweet Sends Daimler, BMW Shares Careening
Bloomberg, 22/06/2018



Pound plunges after Leave vote
BBC, 24/06/2016



Relevant questions (e.g. for an investor):

- » How much money can I lose from one day to the other if I am invested in a specific asset?
- » Are bit coins really 'riskier' than (say) shares of Deutsche Telekom or BMW ?
- » Should I invest in gold instead ?
- »

Market risk : potential losses in the value of financial instruments / assets due to (stochastic) changes in the market conditions

Market risk is an important factor for many financial institutions

- » The market variables driving the changes of the asset prices are called **risk factors**.
- » The **correct identification of all relevant risk factors** is an important step in the development of market risk models
- » **Examples** for risk factors: (stock) prices, exchange rates, interest rates
- » **Measuring market risk** is important for banks (and indeed other institution) for different reasons:
 - › actual management / steering of a bank (investment decisions, risks to take or to avoid,...)
 - › comply with regulation, for example
 - › bank has to understand the risks it takes
 - › equity capital to be kept is determined by the amount of risk the bank is taking
- » There are other important **risk categories** which will not be looked at here
 - › credit risk
 - › operational risk
 - › ...

- » The measurement of market risk requires the use **of statistical models**
- » Goal of today will be to present some of the standard models as well as some more advanced approaches

Measurement and management of market risk is key for many financial institutions, in particular banks.



Measures for market risk

General setup and notations

- » We consider a portfolio with value V_p^t at time t , consisting of K different positions of sizes N_i in assets with value S_i^t ($i = 1, \dots, K$). *Example:* A portfolio of 500 Deutsche Telekom shares would have $N_1 = 500$ and S_1^t would denote the price of one Telekom share.
- » Portfolio values and **1-day portfolio P&Ls** (“profit-and-loss”) are then given by

$$V_p^t = \sum_{i=1}^K N_i \cdot S_i^t \qquad PnL_p^{t+1} = V_p^{t+1} - V_p^t$$

- » We are interested in modelling a probability distribution $F_{PnL_p^{t+1}}$ which **forecasts PnL_p^{t+1} on day t** . The corresponding density is denoted by $f_{PnL_p^{t+1}}$

- » We consider the **Value-at-Risk**

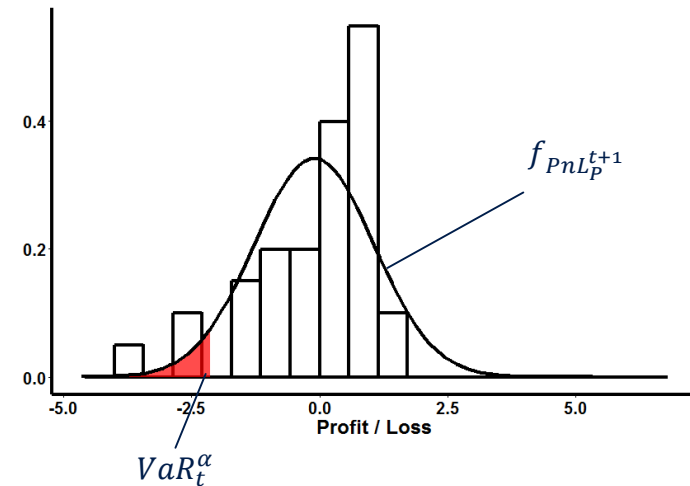
$$VaR_t^\alpha = -\sup(x \in R, \int_x^\infty f_{PnL_p^{t+1}}(x) dx \geq \alpha)$$

with $\alpha = 99\%$.

- » An alternative would be the **Expected Shortfall**

$$ES_t^\alpha = \frac{1}{1-\alpha} \cdot \int_\alpha^1 VaR_t^\beta d\beta$$

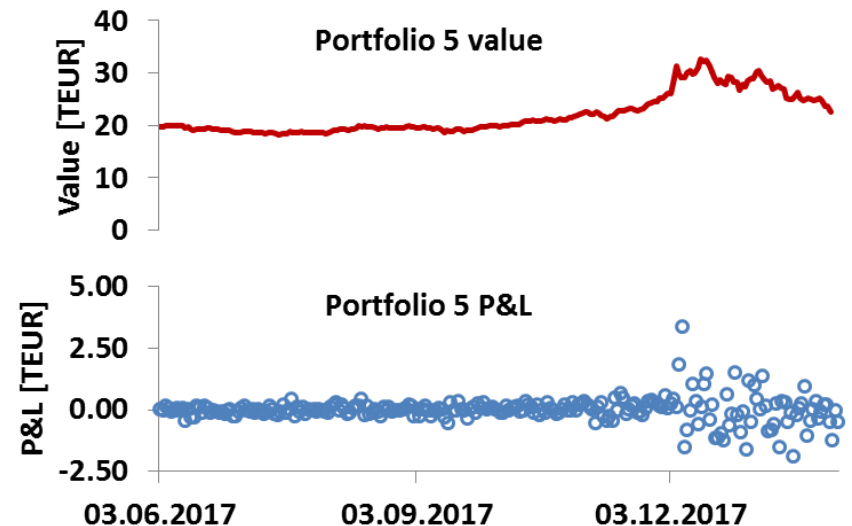
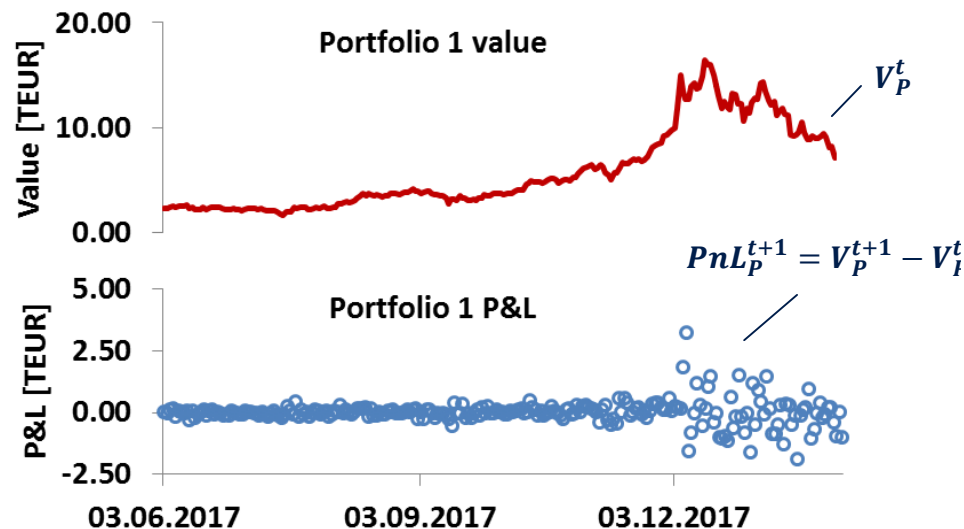
which quantifies the expected loss beyond VaR.



The goal of a market risk model is to forecast the **profit-and-loss probability distribution**, which contains all needed information. The information is usually condensed into a **risk measure**.

Example portfolios

- » Market risk is a **portfolio-specific quantity**. In particular it is in general not the sum of the market risk of the portfolio constituents.
- » In the examples, we will consider portfolios consisting of positions in different assets. In particular the following portfolios will be frequently looked at for illustration
 - » **Portfolio 1:** One Bitcoin (BTC)
 - » **Portfolio 2:** 500 Shares of DTE (Deutsche Telekom)
 - » **Portfolio 3:** 200 shares of DBK (Deutsche Bank)
 - » **Portfolio 4:** 5 ounces of fine gold
 - » **Portfolio 5:** All assets of portfolios 1-4 in one portfolio
- » **Examples:** In the time between 01.06.2017 and 01.01.2018 value V_P^t and P&L PnL_P^{t+1} of portfolios 1 and 5 look like this



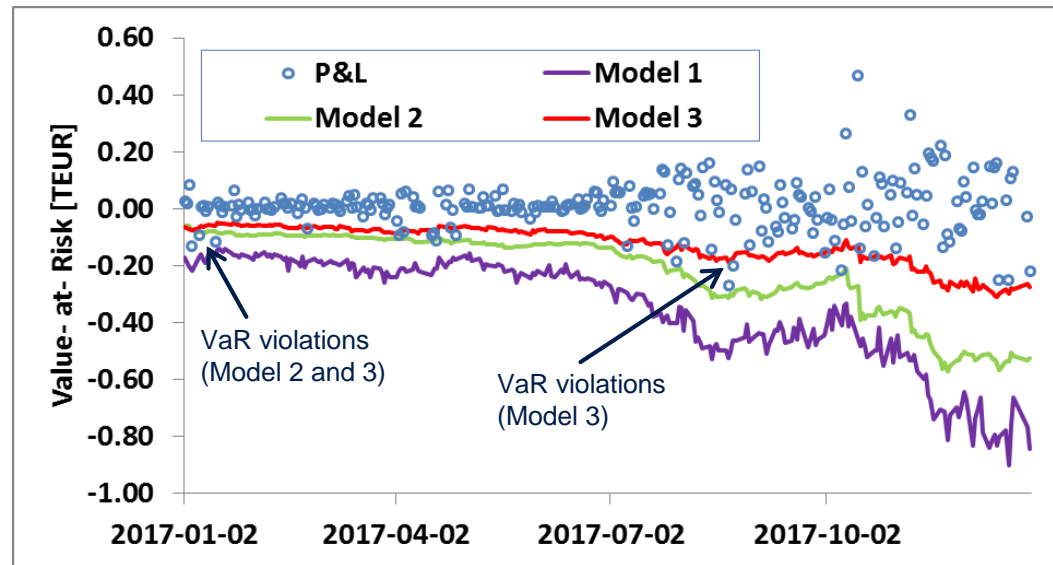
Backtesting – assessing the quality of a market risk model

Assessment of market risk models – examples

- » The quality of different models is assessed via **backtesting**, i.e. the comparison of the models risk forecast with actual market movements

The picture shows risk figures for three different VaR-99% models for portfolio 4

- » Backtesting period is 01.01.2017 – 31.12.2017, which amounts to **250 data points**
- » Blue circles denote the actual portfolio movements
- » The three colored lines show the risk figures computed at the time
- » Expected number of VaR-violations is $250 \cdot 0.01 = 2.5$, so about 2 – 3 violations



Model 1:

- » Modell shows no VaR-violations at all
- » Fewer VaR-violations than expected - model (overly ?) conservative

Model 2:

- » Modell shows 4 VaR – violations
- » These are a few more violations than one would expect - is the model still acceptable ?
- » Some VaR- violations due to statistical fluctuations ?

Model 3:

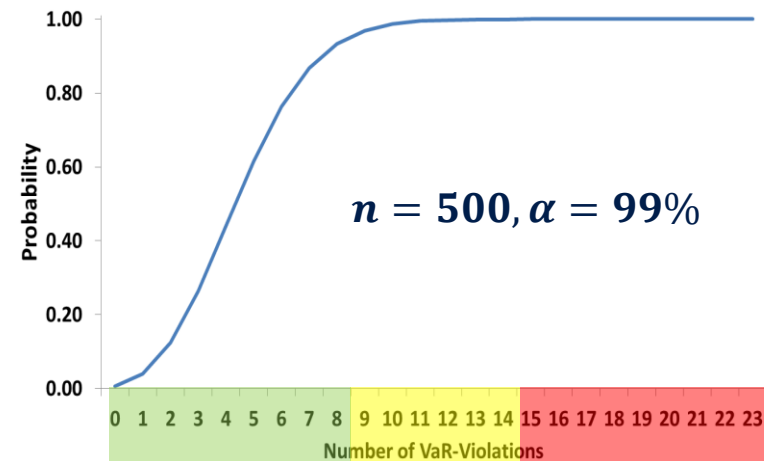
- » Modell shows VaR – violations for 15 days
- » These are fare more violations than one would expect
- » Model should be rejected with high confidence

The quality of market risk models can be assessed by comparing them to realized price movements

Assessment of market risk models – backtesting (1/2)

- » Mathematically this idea can be implemented by applying **statistical tests**, which may allow to reject a model according to defined confidence level.
- » **Backtest 1: Basel Traffic Lights**
 - › One – sided test based on binomial distribution.
 - › If model is correct: Probability of r or less VaR-violations for n days of observation given by $P_r = \sum_{i=0, \dots, r} B_{n, \alpha}(i)$ with the binomial distribution $B_{n, \alpha} = \binom{n}{r} \cdot (1 - \alpha)^r \cdot \alpha^{n-r}$
 - › Fix confidence level p **of the test** and determine maximal r so that $P_{\leq r} \leq p$.
 - › If there are more than r VaR-violations the model can be rejected with confidence level p
- » **Traffic Lights:** defined as function of p
 - › $p \leq 95\%$: Green zone
 - › $95\% < p \leq 99.99\%$: Yellow zone
 - › $p > 99.99\%$: Red zone
- » For an observation period of 2 years ($n = 500$) and a Value-at-Risk confidence level of $\alpha = 99\%$

VaR-Violations	Traffic Light
0-8	green
9-14	yellow
15 or more	red



Assessment of market risk models – backtesting (2/2)

» **Backtest 2: Kupiec Test**

- › Two sided test. Also based on binomial distribution but evaluated via maximum likelihood estimation
- › Very similar to Basel Traffic Lights but test also detects models which are too conservative
- › Compute maximum likelihood function of the binomial distribution $LR_{POF} = -2 \ln \left(\frac{(1-\alpha)^{n-r} \alpha^r}{\left(\frac{1-r}{n}\right)^{n-r} \left(\frac{r}{n}\right)^r} \right)$
- › This converges against a $\chi^2(1)$ distribution for large n
- › Test is considered to have failed if $LR_{POF} > F_{\chi^2}(p)$, where F_{χ^2} is a $\chi^2(1)$ distribution function

» Following the same logic as before, one could define Traffic Lights for $n = 500$ and $\alpha = 0.99$

VaR-Violations	Traffic Light (Kupiec)
2 - 9	green
0-1 or 10-15	yellow
16 or more	red

» But it is more common to consider ‘non-rejection’ intervals for the different p-values. For example for $n = 500$ and $\alpha = 0.99$. **In what follows we will typically use p = 95%**

p-value	95%	99%	99.99%
non-rejection interval	2-9	1-11	0-15

There are many more tests available – we will restrict ourselves to these two tests

Current market standards

Monte Carlo simulation and historical simulation



General setup for simulations

Overview: simulation methods

- » Generally speaking simulation methods try to approximate the P&L distribution by a set of scenarios.
- » The art is to generate appropriate scenarios which reflect market conditions as well as possible

Academic work / publications

- » Markovic, Roy (1952)
- » ...
- » Filtered historical simulation (1998)
- » Barone-Adesi et.al.,
- » Hull et. al.
- » Regulation (Basel 2,...)
- » Regulators acknowledge role of VaR
- » ...

1990

2000

2010

Industry

90s: JP Morgan / Risk Metrics

- » Introduction of VaR as risk measure into fin. industry
- » Methodology published in 1994

90s – 00s – Standard simulation methods

- » Monte Carlo
- » Historical simulation

Advanced methods

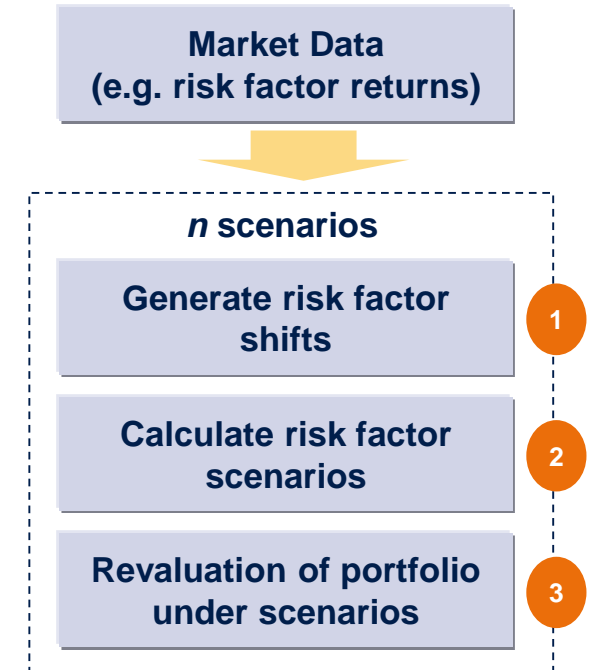
- » Filtered historical simulation
- » ...

- » The increasing computing power and increase in availability of data made simulation methods more and more attractive in the past decade
- » As we will see, modern methods are statistically much more accurate but also more demanding with respect to computational power and the quality and quantity of input data.

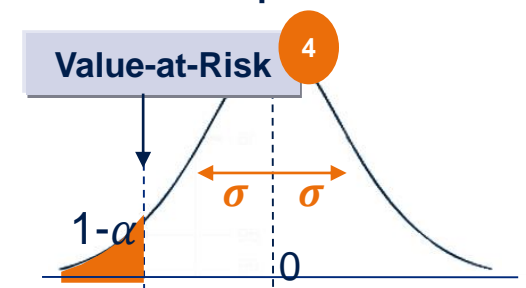
Simulation methods have become more and more accurate in recent years

Standard approaches – general setup for simulations

- » Simulation approaches have the following common setup:
 - » The portfolio consists of several positions. The value of each position depends on **valuation factors** (usually market data). This dependency may be complex, for instance for interest rate curves.
 - » A subset of these valuation factors are selected as **risk factors**, ideally all main risk drivers.
- » Then the computational flow of a typical market risk simulation looks as follows:
 - 1 For each of the risk factor k a large number of **shifts** r_k^i ($i = 1, \dots, n$) is generated which represent potential risk factor evolutions. The shifts should be consistent across risk factors (in a 'correlation' sense)
 - 2 Shifts are applied to current values $S_{t,k}$ of risk factors yielding **risk factor scenarios**, i.e. $S_{t,k}^i = S_t \cdot \exp(r_k^i)$ ($i = 1, \dots, n$)
 - 3 The portfolio is revaluated under each risk factor scenario resulting into a set of portfolio Profits/Losses PnL_t^i ($i = 1, \dots, n$) which represent the **forecasted PnL-distribution**
 - 4 Usually a **risk measure** (e.g. Value-at-Risk) is computed out of this simulated PnL distribution. **For Value-at-Risk:** This amounts essentially to sorting the PnL vector and pick the $(1 - \alpha) \cdot n^{th}$ worst scenario (e.g. for 1000 scenarios and $\alpha = 99\%$ the 10^{th} worst scenario)



Distribution of portfolio PnLs



d-fine

This lecture is mainly about step 1 !



Historical simulation and Monte Carlo simulation

Methods to generate shifts - historical simulation

» The **basic algorithm** to generate scenarios works as follows

1. Pick historical 1-day risk factor returns from a rolling window of size n . Return types can be chosen and often depend on the type of risk factor used

$$r_j^{t,log} := \ln\left(\frac{S_j^t}{S_j^{t-1}}\right) \quad r_j^{t,rel} := \frac{S_j^t - S_j^{t-1}}{S_j^{t-1}} \quad r_j^{t,abs} := S_j^t - S_j^{t-1} \quad t = T, \dots, T - n + 1$$

2. Apply the returns to the current level of the risk factors to obtain risk factor scenarios ($k = 1, \dots, n$)

$$S_j^{T,k} = S_j^T \cdot \exp\left(r_j^{t-k+1,log}\right) = S_j^T \cdot \frac{S_j^{T-k+1}}{S_j^{T-k}}, \quad S_j^{T,k} = S_j^T \cdot \left(1 + r_j^{T-k+1,rel}\right) = S_j^T \cdot \frac{S_j^{t-k+1}}{S_j^{t-k}},$$
$$S_j^{T,k} = S_j^T + r^{T-k+1,abs} = S_j^T + S_j^{T-k+1} - S_j^{t-k}$$

3. Continue as described above

» The most important parameters are:

- » The **number of scenarios** n : should be chosen large but is usually limited by data availability
- » The **return type**: note that for this variant of historical simulation log and relative returns are equivalent (we will usually be using log-returns)

Advantages:

- » No explicit distribution assumption or calibration
- » Simple and transparent
- » Few parameters (can also be a drawback ...)

Drawbacks:

- » Number of scenarios limited by data availability
- » Needs very good data quality

Methods to generate shifts – Monte Carlo

» The **basic algorithm** to generate scenarios works as follows

1. Pick historical 1-day risk factor returns from a rolling window of size K .
2. Calibrate the covariance matrix \mathbf{C}^t to the return history, i.e. compute

$$C_{ij}^t = \frac{1}{K} \sum_{k=0}^{K-1} (r_i^{t-k} - \mu_i^t) \cdot (r_j^{t-k} - \mu_j^t), \quad \mu_i^t = \frac{1}{K} \sum_{k=0}^{K-1} r_i^{t-k}$$

3. Compute the Cholesky decomposition of the covariance matrix $\mathbf{C}^t = \mathbf{A}^t \cdot (\mathbf{A}^t)'$. ($t = \text{time}$, $\mathbf{A}' = \text{transpose of } \mathbf{A}$)
4. Generate independent normal random variables ξ_i^r ($r = 1, \dots, n$) and use the Cholesky decomposition to transform them to correlated normal random variables $r_i^{T,r} = \sum_{j \in J} A_{ij}^T \cdot \xi_j^r$. These are the shifts.
5. Apply the shifts to the current level of the risk factors. For example for log returns
$$S_i^{T,r} = S_i^T \cdot \exp(r_i^{T,r})$$
6. Continue as described above

» The most important parameters are:

- › The **number of scenarios** n : Can – in principle – be chosen arbitrarily large
- › The **return type**
- › The **calibration window size** K for the covariance matrix. Note that often \mathbf{C}^t is not recalibrated daily

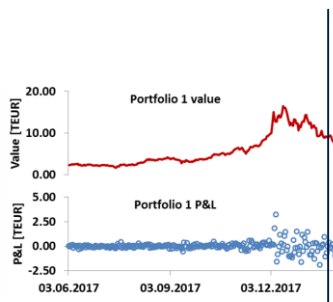
Advantages:

- » Number of scenarios unlimited
- » More robust with respect to data quality

Drawbacks:

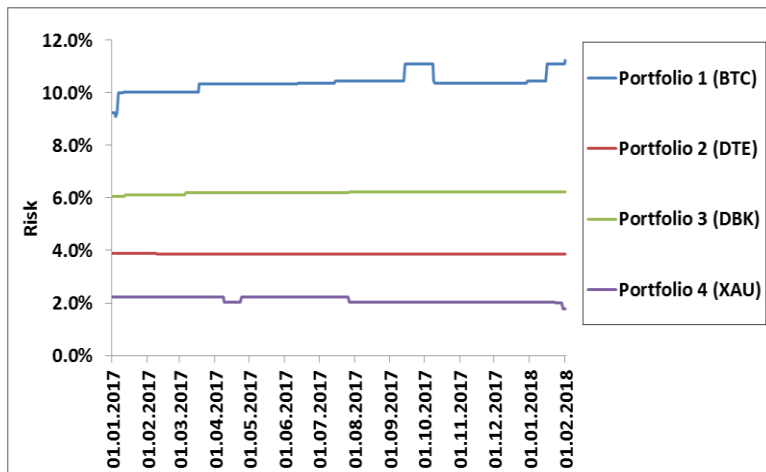
- » More difficult to justify (distribution assumptions etc.) and explain
- » Covariance matrix estimation can be tricky

Example portfolios – result overview



Date	Market Value [TEUR]	Value - at - Risk [TEUR]		Value-at-Risk / Market Value	
		Hist. Sim.	Monte Carlo	Hist. Sim.	Monte Carlo
31.01.2018					
Portfolio 1 (BTC)	8.15	0.91	1.04	11.1%	12.8%
Portfolio 2 (DTE)	7.06	0.27	0.12	3.9%	1.7%
Portfolio 3 (DBK)	2.96	0.18	0.10	6.2%	3.3%
Portfolio 4 (XAU)	5.40	0.10	0.05	1.8%	1.0%
Portfolio 5	23.57	0.99	1.04	4.2%	4.4%

Risk of the example portfolios (1-day, 99% Value-at-Risk) for one valuation day



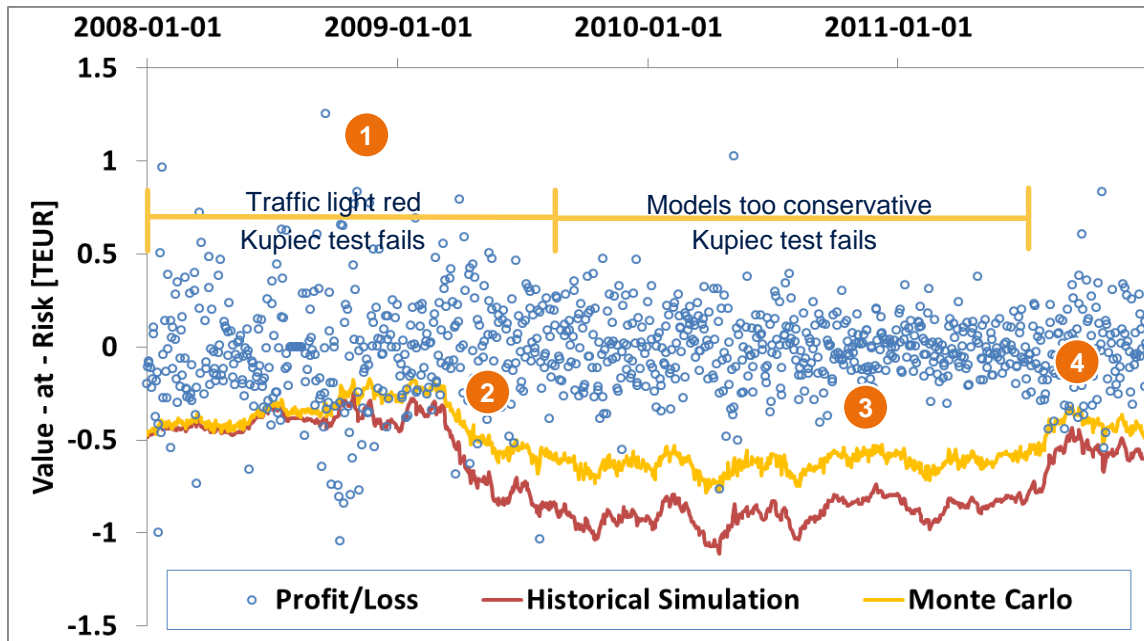
Risk over time relative to asset price for a 1-day, 99% Value-at-Risk, computed by Historical Simulation using 1000 scenarios

- » Using the explained standard methodologies the risk for the example portfolio is computed using
 - » Historical simulation with 1000 scenarios
 - » Monte Carlo simulation with 5000 scenarios (assuming a Gaussian distribution)
- » The risk of the different assets relative to each other is roughly in line with what one would intuitively expect (bit coin riskier than stock, stock riskier than gold)
- » However, results depend strongly on chosen methodology

The used methodology (and its parameterisation) has a strong impact on the estimation of market risk

Weaknesses of standard approaches

» Example portfolio 3 (DBK) from 2008 to 2011:



» **Setup:**

- › Historical simulation with 1000 scenarios (=4 years moving window)
- › Monte Carlo with 2000 scenarios and 4 years calibration window
- › Confidence Level = 99%

» Jan 2008 – Aug 2009: Both backtests fail due to too many outliers.

pf3: Jan 2008 - Aug 2009			
Window Size	#outlier	result Basel	result Kupiec
Hist. Sim.	31	red	failed
Monte Carlo	41	red	failed

» Sep - 2009 to June 2011 Kupiec test fails because model is too conservative

pf3: Sep 2009 - Jun 2011			
Window Size	#outlier	result Basel	result Kupiec
Hist. Sim.	0	green	failed
Monte Carlo	0	green	failed

- 1 Market volatility increases (Lehman-crisis). Large return enter the moving time window but the Value-at-Risk does not 'see' them yet
- 2 Sufficiently many large returns have entered the moving time window so that the risk figures start rising
- 3 Although markets have calmed down the large returns still dominate in the moving time window so risk figures remain on a high level
- 4 Large returns start dropping out of the moving time window and risk figures decrease

Standard historical simulation and Monte Carlo suffer from low reactivity to changing market conditions

Advanced simulation methods

Improving standard methods - overview

- » There are different reasons for the problems encountered above. The main issues are the following:

Historical simulation – main problems

» Lack of reactivity

- › Model does not react to new market conditions
- › Simple remedies like shortening the time window (= fewer scenarios) lead to problems with statistical accuracy
- › *Therefore:* structural improvement needed

Monte Carlo simulation – main problems

» Lack of reactivity

- › Model does not react to new market conditions
- › Estimation technique for covariance matrix needs to be improved

» Lack of fat tails

- › Assumption of Gaussian distribution is wrong
- › Different distribution assumption needed

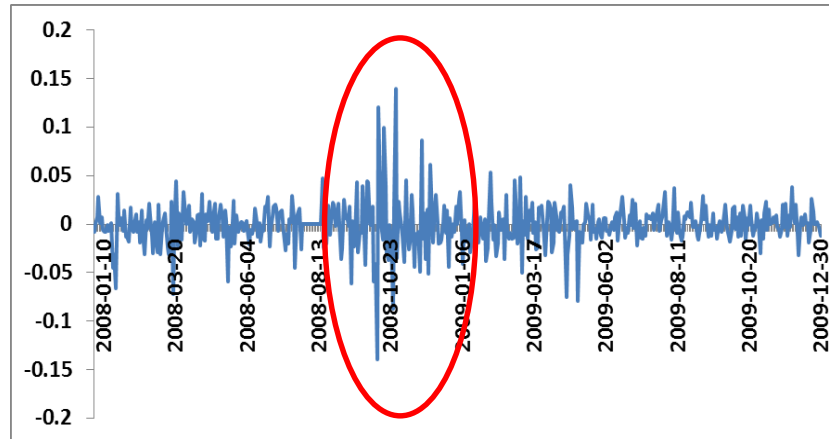
- » Fixing these deficiencies will lead to more sophisticated approaches, which are however even more dependent on the quantity and quality of available data

In order to fix these problems we need to take a look at the structure of the underlying time series

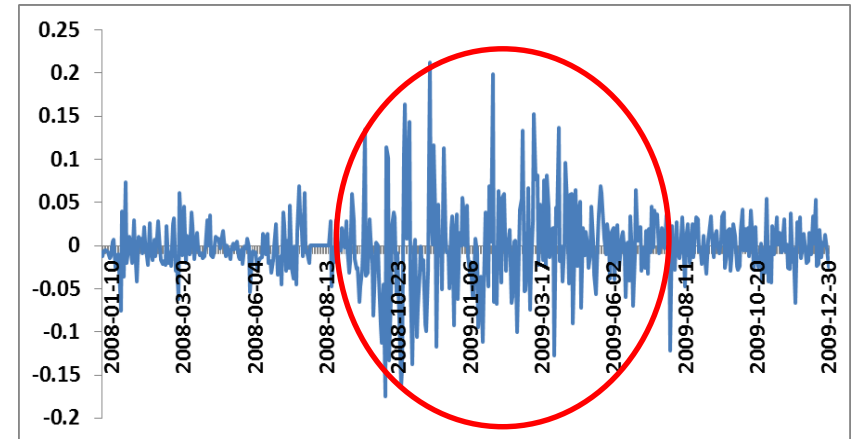
Advanced variants of historical simulation

Filtered historical simulation – motivation

- » In underlying assumption of standard historical simulation is that scenarios are stationary, i.e. each scenario “has the same probability”.



DTE (portfolio 2): Returns from 2008-01-01 to 2009 – 12- 31



DBK (portfolio 3): Returns from 2008-01-01 to 2009 – 12- 31

- » Visual inspection already shows that this assumption is violated, especially if there is a major crisis in the considered time window
- » Filtered historical simulation⁽¹⁾ tries to heal this by introducing a suitable model for the historical volatility σ_j^t (could be EWMA, GARCH ... more on this later) and possibly the drift μ_j^t which are then used to normalise the returns⁽²⁾ r_j^t before they are further processed.

⁽¹⁾ Barone-Adesi, G., Bourgoin, F. and K. Giannopoulos, 1998, *Don't Look Back*, Risk, 11, August

Hull, J. and A. White, 1998, *Incorporating Volatility Updating into the Historical Simulation Method for VaR*, Journal of Risk 1

⁽²⁾ from now on “returns” will always mean “log-returns” unless stated otherwise, that is $r_j^t = r_j^{t,log}$

Filtered historical simulation – general idea

» **Simplest version:** (Univariate) filtered historical simulation with EWMA – volatilities:

1. Compute historical (EWMA-) volatility for each risk factor over time (this can be done recursively). There is one additional parameter $\lambda \in [0,1)$ (per risk factor), a typical value is $\lambda = 0.97$

$$\mu_j^t = (1 - \lambda) \cdot \sum_{i=0}^{\infty} \lambda^i \cdot r_j^{t-i} \quad \sigma_j^t = (1 - \lambda) \cdot \sum_{i=0}^{\infty} \lambda^i \cdot (r_j^{t-i} - \mu_j^{t-i})^2$$

2. Compute **residuals** by dividing each return by its volatility at the same point in time

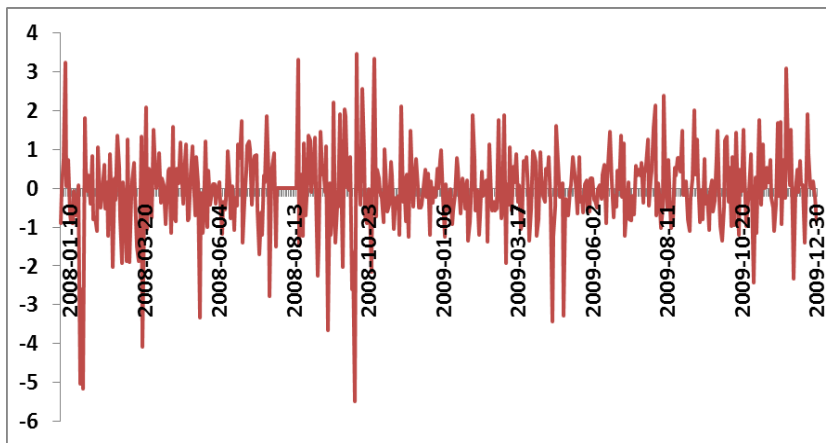
$$\xi_j^t = \frac{r_j^t}{\sigma_j^t}$$

3. At the valuation date T , compute shifts $R_j^{T,k}$ by upscaling all residuals with the volatility at time T

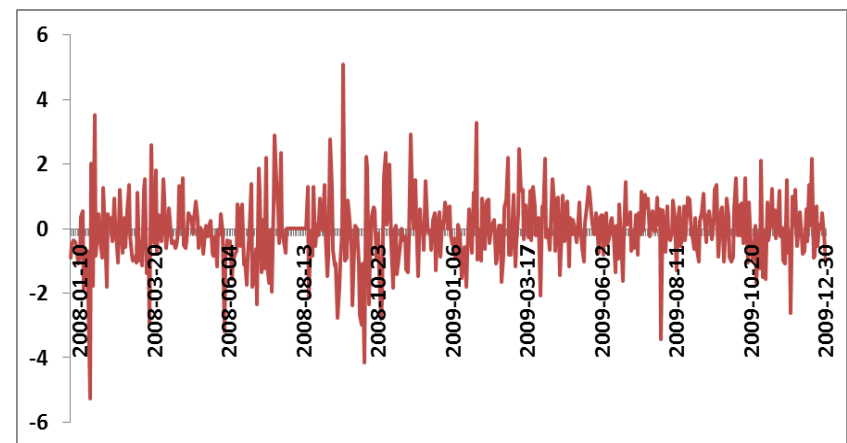
$$R_j^{T,k} = \sigma_j^T \cdot \xi_j^{T-k} \quad (k = 1, \dots, n)$$

4. Use these shifts instead of the original returns in the historical simulation algorithm

» Many different volatility models are available \Rightarrow more on this later



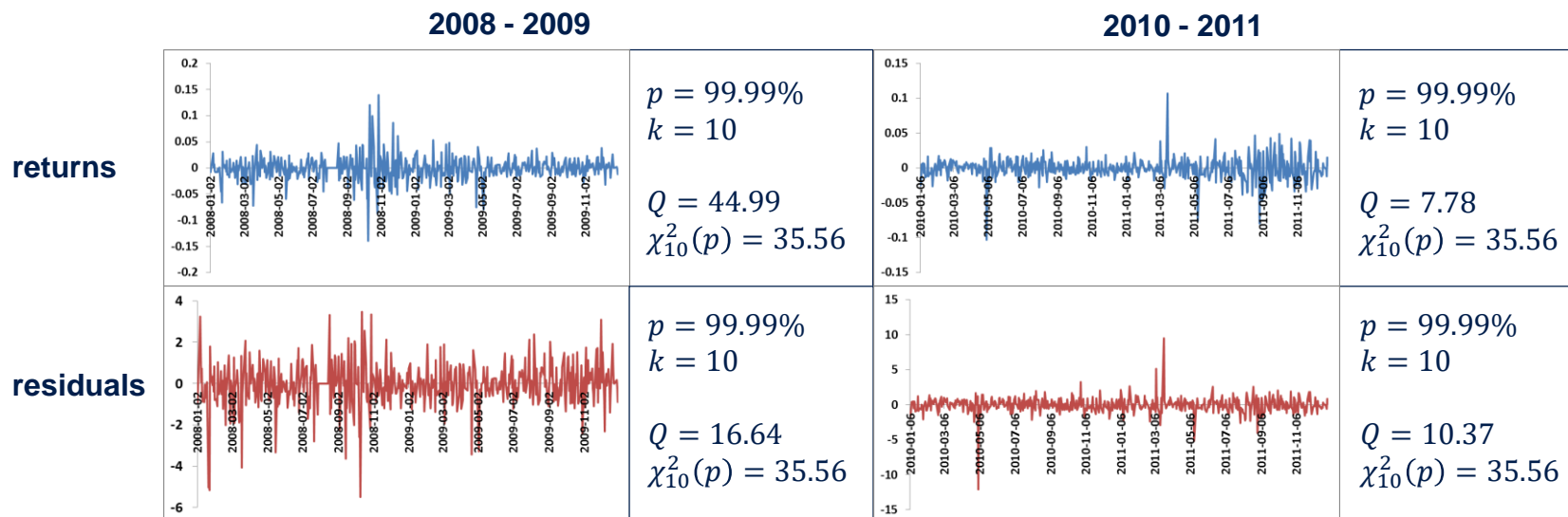
DTE (portfolio 2): Residuals from 2008-01-01 to 2009 – 12- 31



DBK (portfolio 3): Residuals from 2008-01-01 to 2009 – 12- 31

Stationarity: Ljung – Box test

- » Stationarity can be assessed with the **Ljung-Box test**⁽¹⁾
- » The test statistic is given by $Q = n(n + 2) \sum_{k=1}^h \frac{\rho_k^2}{n-k}$, where n is the sample size, ρ_k^2 the standard autocorrelation estimator with lag k and h the number of lags being tested.
- » Null hypothesis is that the sample **is independently distributed**. It can be rejected with conf. p if $Q > \chi_h^2(p)$ where $\chi_h^2(p)$ denotes the p -quantile of a χ^2 distribution with h degrees of freedom.
- » **Example:** Returns and residuals of DTE share (portfolio 2) in 2008-2009 and 2010-2011

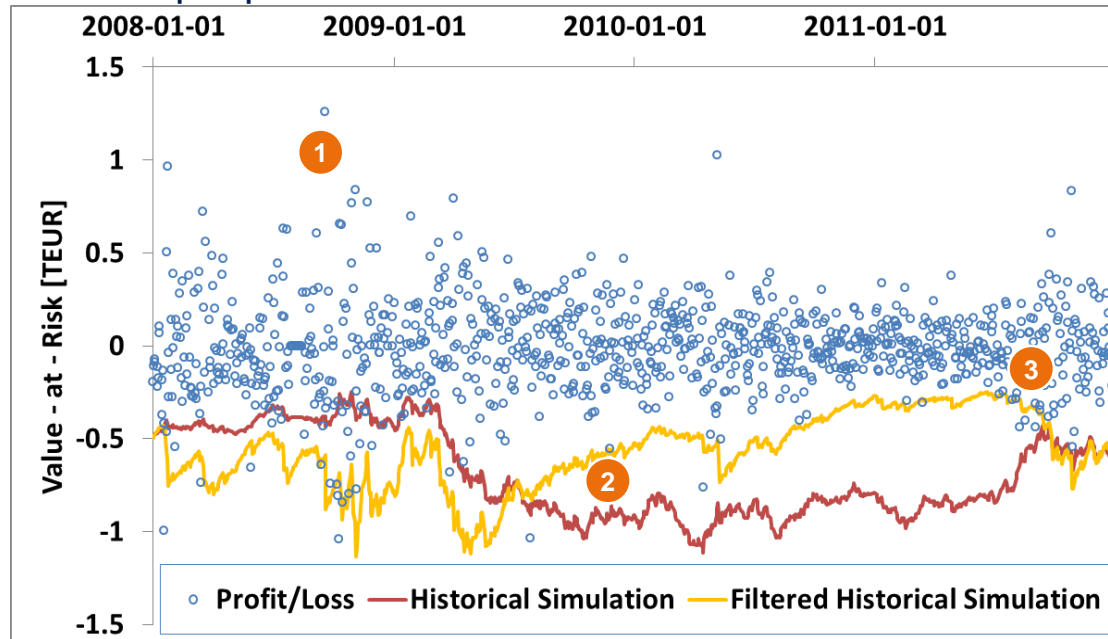


⁽¹⁾ G. M. Ljung, G.E.P. Box (1978), *On a Measure of a Lack of Fit in Time Series Models*, Biometrika 65 (2), 297 - 303

Residuals are stationary also in times of crisis, returns only in calm periods

Filtered historical simulation - backtesting

» Example portfolio 3 from 2008 to 2011:



- 1 Market volatility increases (Lehman-crisis). The EWMA – volatility in the FHS picks up, shifts in the simulation become larger and VaR increases
- 2 Market volatility decreases. Due to the lower upscaling factor shifts in the FHS become smaller and VaR decreases
- 3 The Euro-crisis leads again to increasing volatility in the market to which the FHS reacts swiftly again.

» Setup:

- › Historical simulation with 1000 Szenarios (=4 years moving window)
- › Filtered historical simulation with 1000 Szenarios and decay factor $\lambda = 0.97$
- › Confidence Level = 99%

» Jan 2008 – Aug 2009: Number of outliers significantly reduced for FHS

pf3: Jan 2008 - Aug 2009			
Model	#outlier	result Basel	result Kupiec
HS	31	red	failed
FHS	9	yellow	failed

» Sep 2009 - June 2011:

pf3: Sep 2009 - Jun 2011			
Model	#outlier	result Basel	result Kupiec
HS	0	green	failed
FHS	2	green	passed

Reactivity and statistics of filtered historical simulation are significantly superior to standard historical simulation

Filtered historical simulation – modelling the historical volatility (1/2)

» Advantages:

- › Significantly enhanced statistical properties, especially in times of crisis
- › New modelling possibilities via link with historical volatility models.

» Disadvantages:

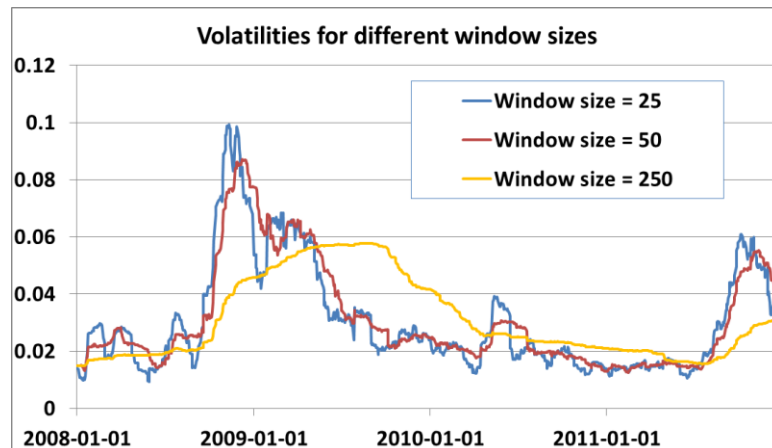
- › Increased complexity, less transparent
- › Calibration and model risk
- › Even longer and better quality time series needed

» The new element of Filtered Historical Simulation is that a (sufficiently reactive or forward-looking) **model for the historical volatility** is needed.

- › On the one hand, this introduces **additional complexity, model risk** and **calibration efforts**.
- › On the other hand it opens the door for new risk modelling aspects and ways to improve the model.

» **Example:** Volatility of DBK (portfolio 3) from 2008 to 2011:

1. Standard volatility estimator with varying window size



$$\mu_j^t(K) = \mu_j^t = \frac{1}{K} \sum_{i=0}^{K-1} r_j^{t-i}$$

$$(\sigma_j^t)^2(K) = (\sigma_j^t)^2 = \frac{1}{K-1} \sum_{i=0}^{K-1} (r_j^{t-i} - \mu_j^{t-i})^2$$

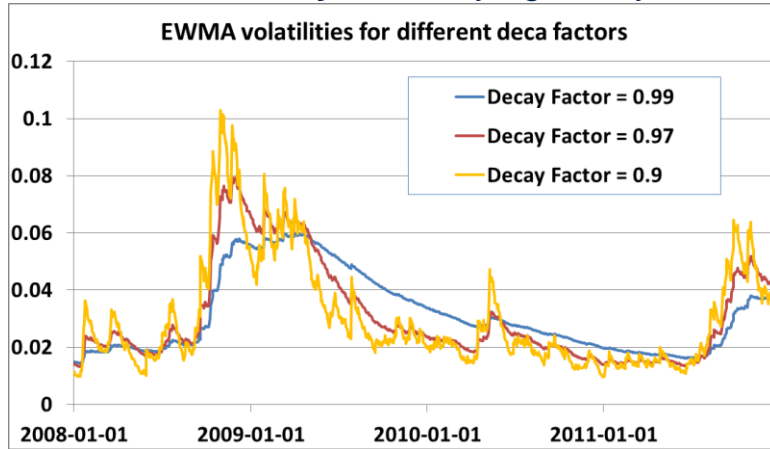
K = Window size, j = risk factor

Reactivity is purely determined by window size of the estimator:

- » Small window \Rightarrow more reactive
- » Large window \Rightarrow less reactive

Filtered historical simulation – modelling the historical volatility (2/2)

2. EWMA – volatility with varying decay factor



$$\mu_j^t = (1 - \lambda) \sum_{i=0}^{\infty} \lambda^i r_j^{t-i}$$

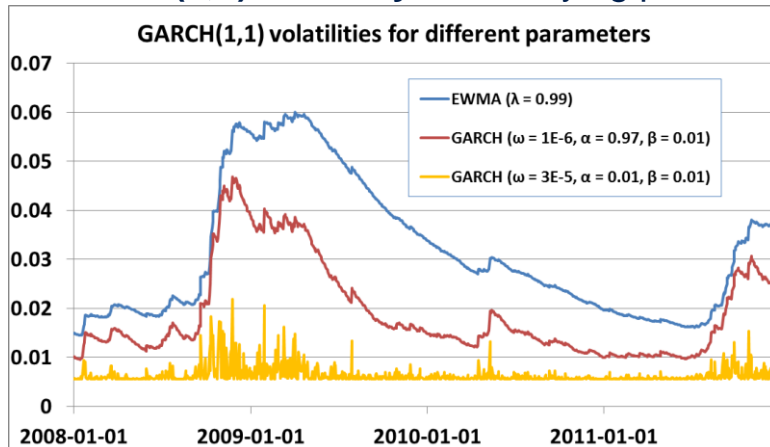
$$(\sigma_j^t)^2 = (1 - \lambda) \sum_{i=0}^{\infty} \lambda^i (r_j^{t-i} - \mu_j^{t-i})^2$$

$\lambda \in [0,1)$ “decay factor” determines the reactivity, can also be computed recursively

$$\mu_j^t = \lambda \mu_j^{t-1} + (1 - \lambda) r_j^t$$

$$(\sigma_j^t)^2 = \lambda (\sigma_j^{t-1})^2 + (1 - \lambda) (r_j^t - \mu_j^t)^2$$

3. GARCH (1,1)-volatility with varying parameters



Recursive computation (assumption $\mu_j^t = 0$)

$$(\sigma_j^t)^2 = \omega + \alpha (\sigma_j^{t-1})^2 + \beta (r_j^t)^2$$

parameters: ω, α, β . Contains EWMA as special case

$$\omega = 0 \quad \alpha = \lambda \quad \beta = (1 - \lambda)$$

Different representation with $\sigma_j^2 = E[(\sigma_j^t)^2]$ (long-term variance)

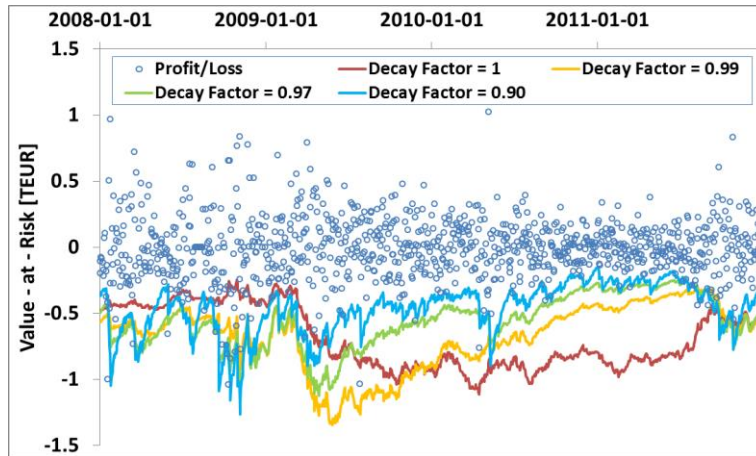
$$(\sigma_j^t)^2 = \sigma_j^2 + \alpha \left((\sigma_j^{t-1})^2 - \sigma_j^2 \right) + \beta \left((r_j^t)^2 - \sigma_j^2 \right)$$

(1) Robert F. Engle: Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of UK. Inflation, Econometrica (50), 1982.

T. Bollerslev: Generalized Autoregressive Conditional Heteroskedasticity, Journal of Econometrics (31), 1986

Filtered historical simulation – further examples

Portfolio 3: FHS with EWMA volatility for different decay factors



» Jan 2008 – Aug 2009:

pf3: Jan 2008 - Aug 2009			
Decay Factor	#outlier	result Basel	result Kupiec
1	31	red	failed
0.99	11	yellow	failed
0.97	9	yellow	failed
0.90	15	red	failed

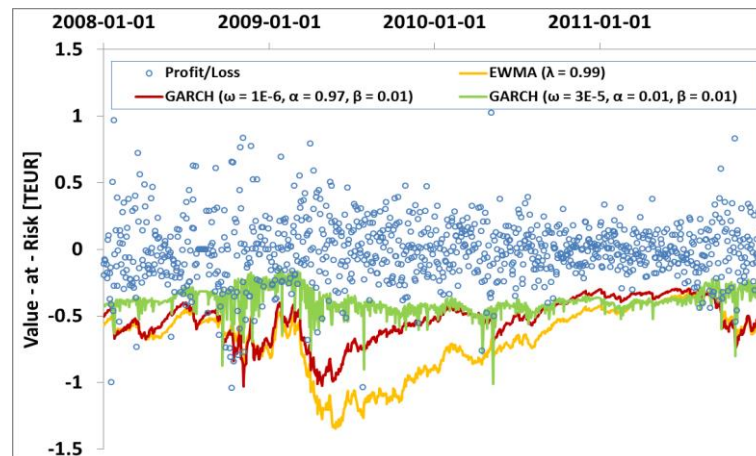
- » filtering yields improvement but test still yellow / failed
- » For very small decay factors performance gets worse

» Sep 2009 - June 2011:

pf3: Sep 2009 - Jun 2011			
Decay Factor	#outlier	result Basel	result Kupiec
1	0	green	failed
0.99	0	green	failed
0.97	2	green	passed
0.90	7	green	passed

- » filtering yields sufficient improvement

Portfolio 3: FHS with GARCH(1,1) volatility for different parameters and with EWMA volatility with decay factor 0.97



» Jan 2008 – Aug 2009:

pf3: Jan 2008 - Aug 2009			
Decay Factor	#outlier	result Basel	result Kupiec
$\lambda = 0.99$	11	yellow	failed
$\omega = 1E-6, \alpha = 0.97, \beta = 0.01$	12	yellow	failed
$\omega = 3E-5, \alpha = 0.01, \beta = 0.01$	44	red	failed

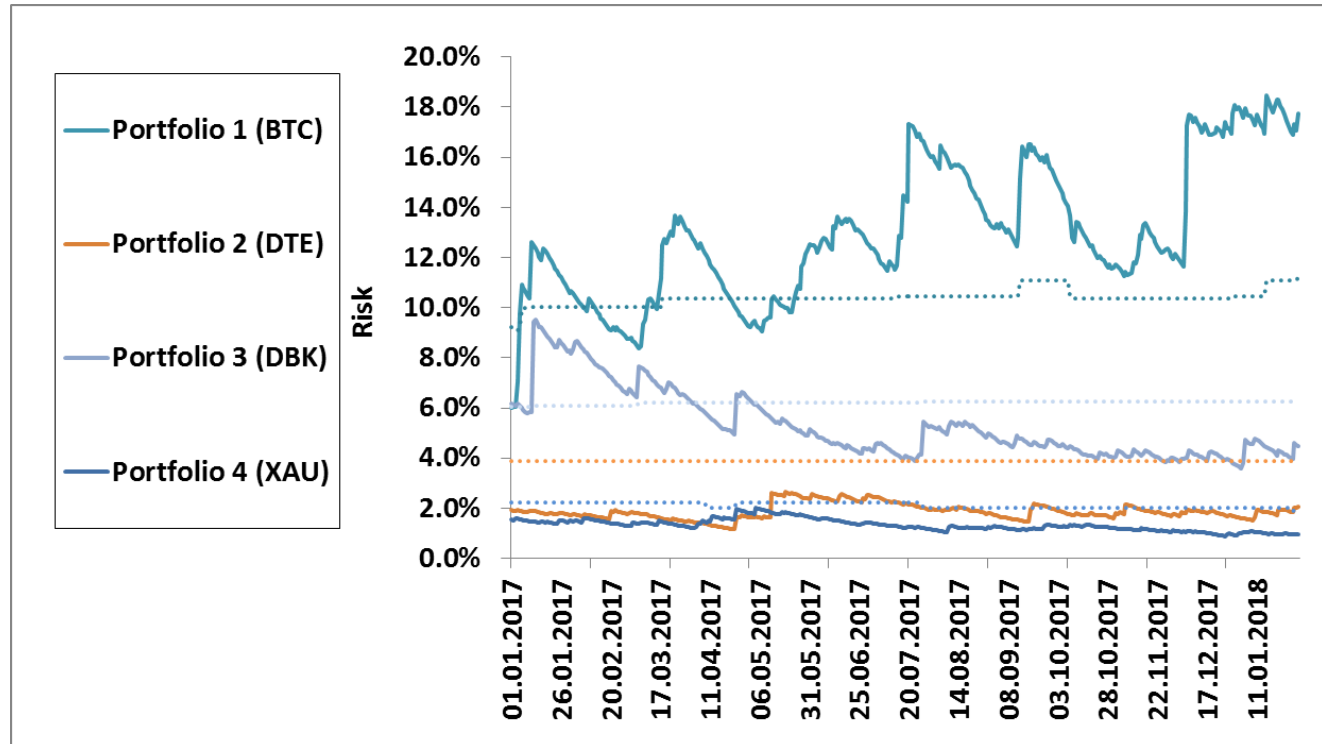
- » Garch(1,1) model not better than EWMA model
- » Wrong parameterisation can lead to very bad results

» Sep 2009 - June 2011:

pf3: Sep 2009 - Jun 2011			
Decay Factor	#outlier	result Basel	result Kupiec
$\lambda = 0.99$	0	green	failed
$\omega = 1E-6, \alpha = 0.97, \beta = 0.01$	2	green	passed
$\omega = 3E-5, \alpha = 0.01, \beta = 0.01$	3	green	passed

- » No significant improvement by using Garch(1,1) instead of EWMA

Example portfolios – result overview (1/2)



Risk over time relative to asset price for a 1-day, 99% Value-at-Risk, computed by Filtered Historical Simulation using 1000 scenarios (decay factor = 0.97). Figures for Standard Historical Simulation are shown as dotted lines

Historical Simulation with and without volatility filtering for the example portfolios

- » Risk figures with and without volatility filter can be significantly different
- » Deviations from ‘filtered’ to ‘unfiltered’ risk figures occur in both directions
- » Filtered historical simulation reveals trends over time much better

Risk figures change more dynamically under a filtered historical simulation

Example portfolios – result overview (2/2)

Date	31.01.2018	Value - at - Risk [TEUR]			Value-at-Risk / Market Value		
	Market Value [TEUR]	Hist. Sim.	Monte Carlo	Filtered Hist. Sim.	Hist. Sim.	Monte Carlo	Filtered Hist. Sim.
Portfolio 1 (BTC)	8.15	0.91	1.04	1.39	11.1%	12.8%	17.1%
Portfolio 2 (DTE)	7.06	0.27	0.12	0.14	3.9%	1.7%	2.0%
Portfolio 3 (DBK)	2.96	0.18	0.10	0.13	6.2%	3.3%	4.5%
Portfolio 4 (XAU)	5.40	0.10	0.05	0.05	1.8%	1.0%	1.0%
Portfolio 5	23.57	0.99	1.04	1.45	4.2%	4.4%	6.2%

- » Using the more sophisticated methodology confirms the order of magnitude of market risk found earlier
- » Nevertheless the actual risk figures differ sometimes quite significantly
- » Some additional information is revealed:
 - › The (relative) market risk of BitCoins shows a clear upward trend, the risk of DBK a downward trend
 - › The concept of gold being less risky than stock becomes less clear (at some point in time it seems riskier than DTE)

Risk figures change more dynamically under a filtered historical simulation

Historical simulation: comparison of approaches

	Standard Historical Simulation	Filtered Historical Simulation
Methodological soundness	+	+
Reactivity / Accuracy	-	+
Complexity	+	-
Transparency	+	○
Dependency on data quality	-	-
Calibration	+	-

Strong improvement ...
... but comes at a cost

longer & better time series needed !

additional data for computation of historical volatility rolling window for scenarios

one 'bad' data point (fat finger trade, technical error, ...) can destroy the whole distribution !

Filtered historical simulation offers a significant improvement of statistical properties but is significantly more complex



Improving Monte Carlo

Monte Carlo – Gaussian distribution (1/2)

- » At the core of a Monte Carlo simulation is the generation of risk factors shifts according to a specific distribution. The simplest case is to assume a (multivariate) Gaussian distribution (density):

$$f_N(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} \cdot |\mathbf{C}|^{\frac{1}{2}}} \cdot \exp\left(-\frac{1}{2} \cdot (\mathbf{x} - \boldsymbol{\mu})^t \cdot \mathbf{C}^{-1} \cdot (\mathbf{x} - \boldsymbol{\mu})\right)$$

- » d : number of risk factors
- » \mathbf{x} : vector of risk factor shifts ($d \times 1$)
- » $\boldsymbol{\mu}$: drift vector ($d \times 1$)
- » \mathbf{C} : Covariance matrix ($d \times d$)
- » $|\mathbf{C}|$: determinant of \mathbf{C}

» Drawbacks:

- » Gaussian distribution has no fat tails, i.e. tends to underestimate risk

» Advantages:

- » Simple and easy to explain
- » Covariance matrix (and drift) fully determines distribution.
- » Estimation of parameters (relatively) simply

» Calibration:

- » Drift vector $\boldsymbol{\mu}$ is typically assumed to be zero
- » Covariance \mathbf{C} matrix needs to be calibrated to historical data (see next slide)

» Generation of Risk Factor Shifts:

- » Decompose covariance matrix such that $\mathbf{C} = \mathbf{A} \cdot \mathbf{A}^t$ (for example Cholesky decomposition⁽¹⁾)
- » Generate independent, identically normally distributed shift vectors \mathbf{y}

- » Correlated shift vectors given by $\mathbf{x} := \mathbf{A} \cdot \mathbf{y}$, as $Cov(x_j, x_k) = E\left(\left(\sum_r A_{jr} y_r\right) \cdot \left(\sum_s A_{ks} y_s\right)\right) = \sum_{r,s} A_{jr} \cdot A_{ks} \underbrace{E(y_r y_s)}_{= \delta_{rs}} = C_{rk}$

⁽¹⁾ H. Press et. al. , Numerical recipes, Cambridge University Press

Monte Carlo – Gaussian distribution (2/2)

- » **Estimation of the covariance matrix:** Most common approach is to use the **maximum likelihood estimator**. Given a time series of (historical) returns \mathbf{x}_i ($i = 1, \dots, n$), one attempts to find the matrix \mathbf{C} which maximises the likelihood function

$$L = (2\pi)^{-\frac{nd}{2}} |\mathbf{C}|^{-\frac{1}{2}} \prod_i \exp\left(-\frac{1}{2}(\mathbf{x}_i - \boldsymbol{\mu})^t \cdot \mathbf{C}^{-1} \cdot (\mathbf{x}_i - \boldsymbol{\mu})\right)$$

- » This leads to the well-known estimator (assuming $\boldsymbol{\mu} = 0$)

$$C_{jk} = \frac{1}{n} \sum_i x_i^{(j)} \cdot x_i^{(k)}$$

- » For a large number of risk factors (i.e. $d > n$) the standard estimator can become numerically troublesome. So-called **shrinkage estimators** try to provide more robust estimations in such cases⁽¹⁾.
- » **Direct sampling from time series:** Assuming the ML estimator is good enough and using the time series \mathbf{x}_i ($i = 1, \dots, n$) to define the $d \times n$ matrix

$$\mathbf{X} = \frac{1}{\sqrt{n}} (\mathbf{x}_1 \quad \dots \quad \mathbf{x}_n)$$

one can rewrite the covariance estimator as

$$\mathbf{C} = \mathbf{X} \cdot \mathbf{X}'$$

- » This allows us to generate correlated normal random variables without explicitly computing the Cholesky decomposition

⁽¹⁾ See for example: O. Ledoit, M. Wolf, *Honey, I Shrank the Sample Covariance Matrix*, 2003

Monte Carlo – improve reactivity

» Improving the reactivity of a Monte Carlo simulation can follow similar lines as for Historical Simulation:

1. Reduce **window size for the calibration of the covariance matrix** . This is simple but has the disadvantage of increasing the statistical error

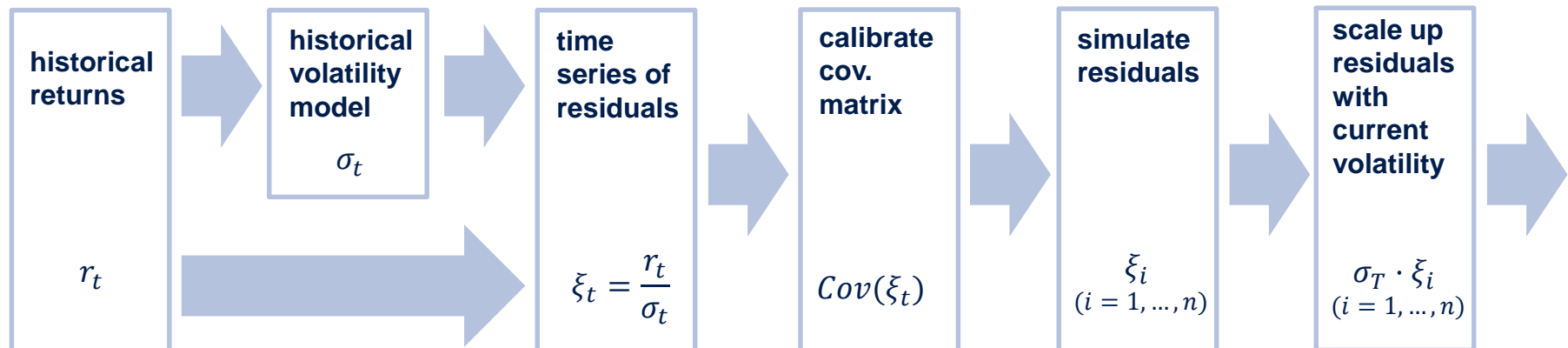
most common approach

2. Use a **weighted estimator** for the covariance matrix, i.e. modify the estimator above as follows

$$C_{jk} = (1 - \lambda) \cdot \sum_i \lambda^i x_i^{(j)} \cdot x_i^{(k)}$$

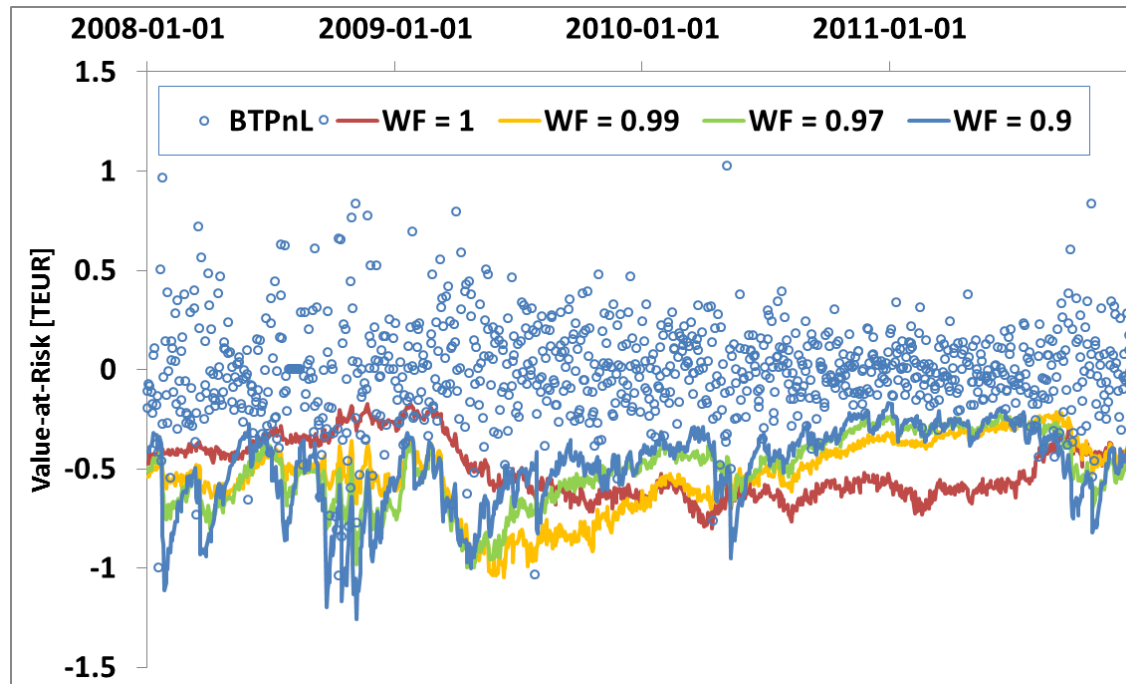
assuming the historical returns x_i ($i = 0, \dots$) are ordered along the time axis (x_0 being the most recent one)

3. Proceed similarly as for filtered historical simulation, i.e. introduce a model for the historical volatility and simulate residuals rather than returns



Monte Carlo – weighted calibration examples (1/2)

» Example Portfolio 3 (DBK) from 2008 to 2011:



» Backtesting results Jan 2008 – August 2009

pf2: Jan 2008 - Aug 2009			
Weighting Factor	#outlier	result Basel	result Kupiec
1	46	red	failed
0.99	13	yellow	failed
0.97	13	yellow	failed
0.90	12	yellow	failed

Backtesting results Sep 2009 – June 2011

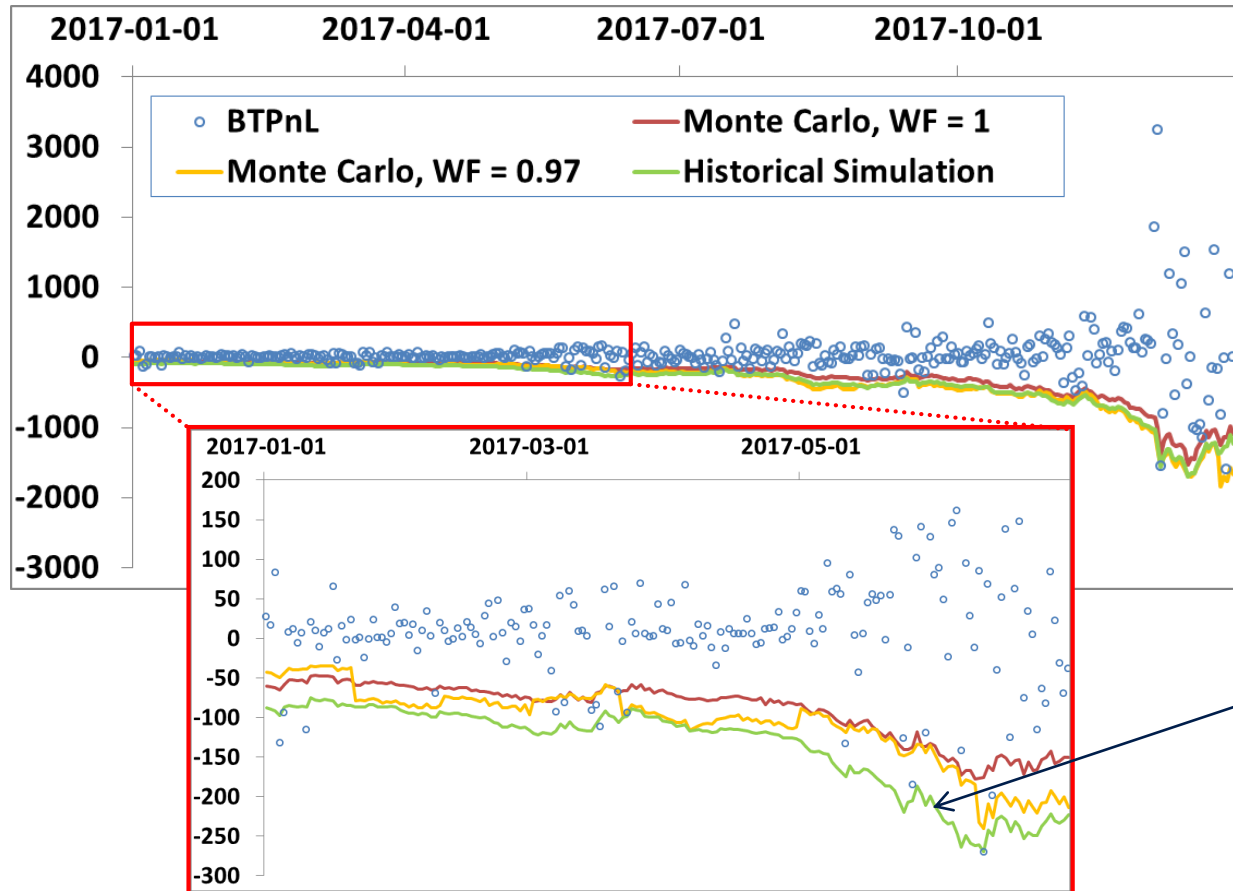
pf2: Sep 2009 - Jun 2011			
Weighting Factor	#outlier	result Basel	result Kupiec
1	1	green	failed
0.99	1	green	passed
0.97	4	green	passed
0.90	7	green	passed

- » Popular method in the market, which for instance vendor risk systems have often implemented
- » Method improves the backtesting results but not sufficiently to pass the backtests

A relatively easy way to increase the reactivity of a Monte Carlo simulation is the use of a weighted calibration of the covariance matrix

Monte Carlo – weighted calibration examples (2/2)

» Example Portfolio 1 (BTC) in 2017:



pf1: Jan 2017 - Dec 2017			
Weighting Factor	#outlier	Basel	Kupiec
Monte Carlo (MC)	18	red	failed
MC & WF = 0.97	15	red	failed

» Increasing the reactivity eliminates some outliers but remaining number of outliers still unacceptably high

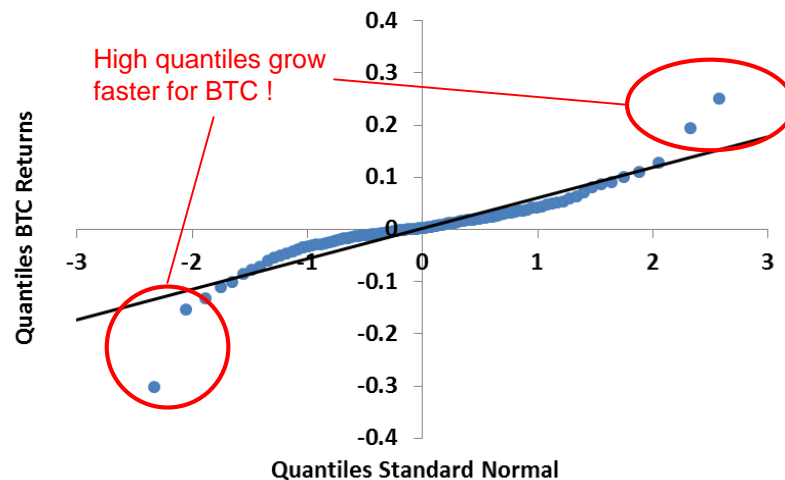
» Comparison with standard historical simulation shows that the Monte Carlo simulation is generally on the wrong level

For Monte Carlo Simulation lack of reactivity is not the only issue but also the distributional assumption can lead to underestimation of risk

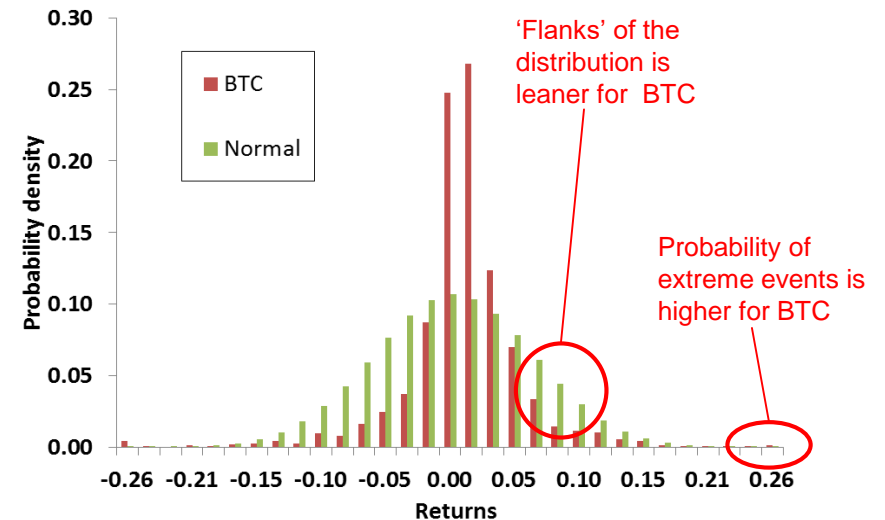
Fat Tails – A common phenomenon in the financial world

- » Consider the example of BTC in Portfolio 1 compared to a normal distribution

Q-Q-plot: Quantiles of two distributions are plotted as x-y diagram



Probability densities: Fat tails mean rare events are more likely



- » The effect just encountered is a manifestation of a more general phenomenon encountered for many return time series in finance: **the existence of 'fat tails'**
- » This poses challenges in (market) risk management, as a risk manager is particularly interested in the 'rare' events (usually the bad ones ...)

Many 'extreme' events in finance occur more often than implied by the Gaussian distribution. **Probability distributions are typically 'fat-tailed'.**

Monte Carlo – Student-t-distribution

- » Major drawback of using the (multivariate) Gaussian distribution is the lack of fat tails
- » The probably simplest fat-tailed distribution available is the **d-dimensional student-t-distribution**, which can be defined by the probability density

$$f_{st}^{\Sigma, k}(\mathbf{x}) = \frac{\Gamma\left(\frac{1}{2}(k+d)\right)}{\Gamma\left(\frac{1}{2}k\right) \cdot (\pi k)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} \cdot \left(1 + \frac{(\mathbf{x} - \boldsymbol{\mu})^t \cdot \Sigma^{-1} \cdot (\mathbf{x} - \boldsymbol{\mu})}{k}\right)^{-(k+d)/2}$$

- » d : number of risk factors
- » k : “degrees of freedom”
- » $\Gamma(\cdot)$: Gamma function
- » \mathbf{x} : vector of risk factor shifts ($d \times 1$)
- » $\boldsymbol{\mu}$: drift vector ($d \times 1$)
- » Σ : positive definite matrix ($d \times d$)
- » $|\Sigma|$: determinant of Σ

» Important properties :

- » The covariance \mathbf{C} matrix of t-distributed random variable and Σ are related by $\mathbf{C} = \frac{k}{k-2} \cdot \Sigma$
- » The expected value of a t-distributed random variable is equal to $\boldsymbol{\mu}$
- » For $k \rightarrow \infty$ a t-distribution converges to a Gaussian distribution
- » For large x the distribution decreases as $f(x) \sim |x|^{-(k+d)}$
- » Excess kurtosis κ_i of the i -th variable (i.e. the standardised fourth moment minus 3: $\kappa_i = E\left(\left(\frac{X-\mu}{\sigma}\right)^4\right) - 3$) is given by

$$\kappa_i = \frac{6}{k-4} \quad (\text{for } k > 4)$$

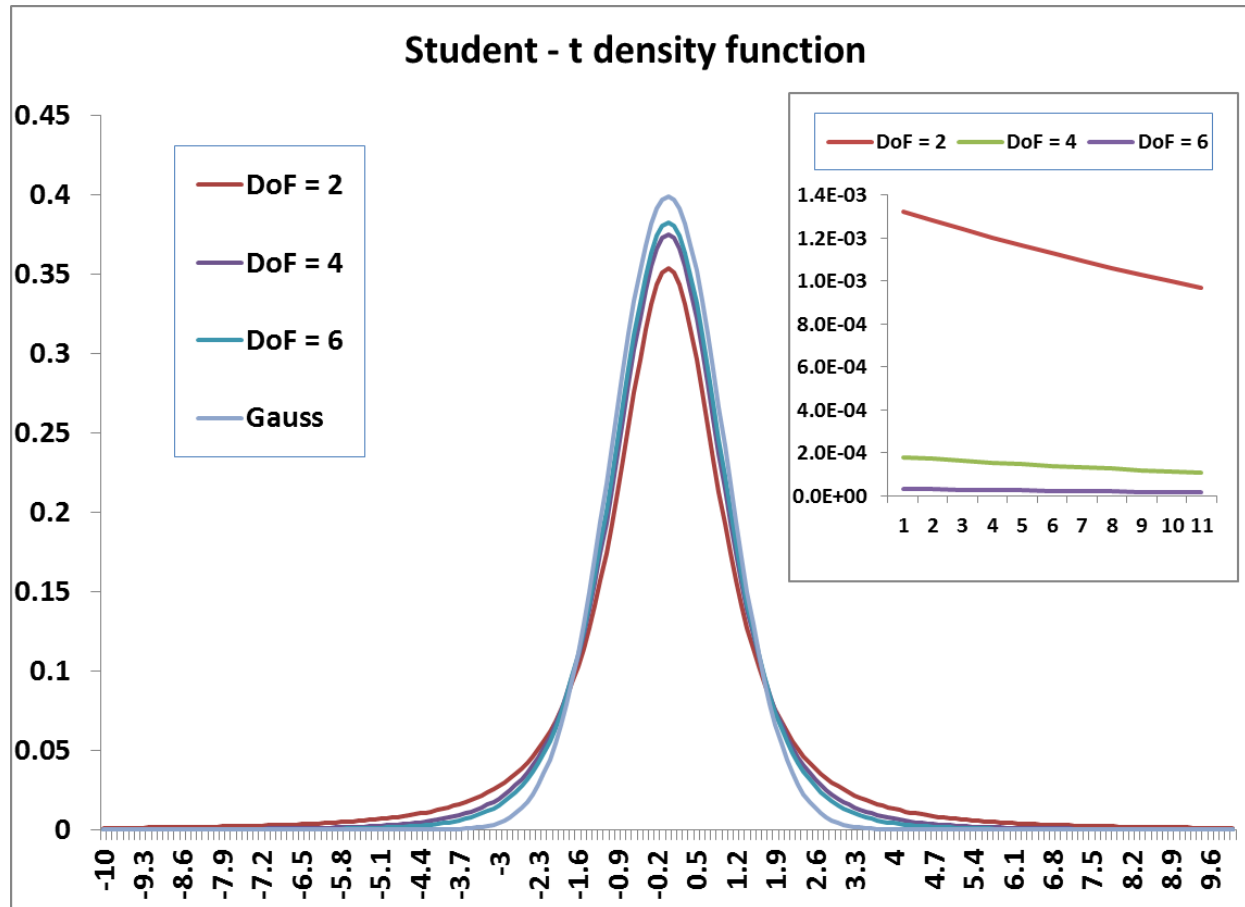
» Drawbacks:

- » More difficult to calibrate
- » Still ‘isotropic’ – one parameter k for all risk factors

» Advantages:

- » Includes fat tails
- » Only one additional parameter needed

Student-t Distribution – dependency on the degrees of freedom



- » The smaller the degree of freedom k the more fat-tailed the distribution becomes
- » For larger k the distributions become more and more similar. Example: 99% quantile for different k

k	99% - Quantile
1	31.82051595
2	6.964556734
3	4.540702859
4	3.746947388
5	3.364929999
10	2.763769458
20	2.527977003
Gauss	2.326347874

- » The second moment exists only for $k > 2$
- » The fourth moment exists only for $k > 4$

The Student-t distribution is the simplest modification of a Gaussian distribution to have fat tails

Calibration

- » The likelihood function does not allow for an explicit solution any more, due to its involved dependency on Σ and k .

$$L(\Sigma, k; \mathbf{x}_1, \dots, \mathbf{x}_n) = \prod_i f_{st}^{\Sigma, k}(\mathbf{x}_i)$$

- » **Full solution:** Optimise the maximum likelihood function numerically. Typically so-called EM – algorithms (**Expectation-Maximisation**) are used which attempt to iteratively find an optimal calibration^(1,2).
 - » Approach is quite generic (also works for other distributions) but relatively complex
 - » Statistical packages often have this implemented. See for example package “ghyp” in R⁽³⁾.
- » **Pragmatic approaches:** Calibrate degrees of freedom and covariance matrix independently
- » Degrees of freedom: Possible approaches
 - » Determine the “tail index” (i.e. look at the asymptotic behaviour for large x) using a known algorithm like the **Hill estimator**:
 - » Compute excess kurtosis’ κ_i for different the different variables via the ‘standard’ estimator and infer degrees of freedom via the relation $k = \frac{6}{\frac{1}{n} \sum_i \kappa_i} + 4$ (“moment matching”)
- » Compute covariance matrix using the ‘standard’ estimator $C_{kj} = \frac{1}{n} \sum_i x_i^{(k)} \cdot x_i^{(j)}$ and the relation $\mathbf{C} = \frac{k}{k-2} \cdot \Sigma$
- » All approaches typically have in common that the degree of freedom fluctuates quite strongly over time (and across risk factors) – see next slide

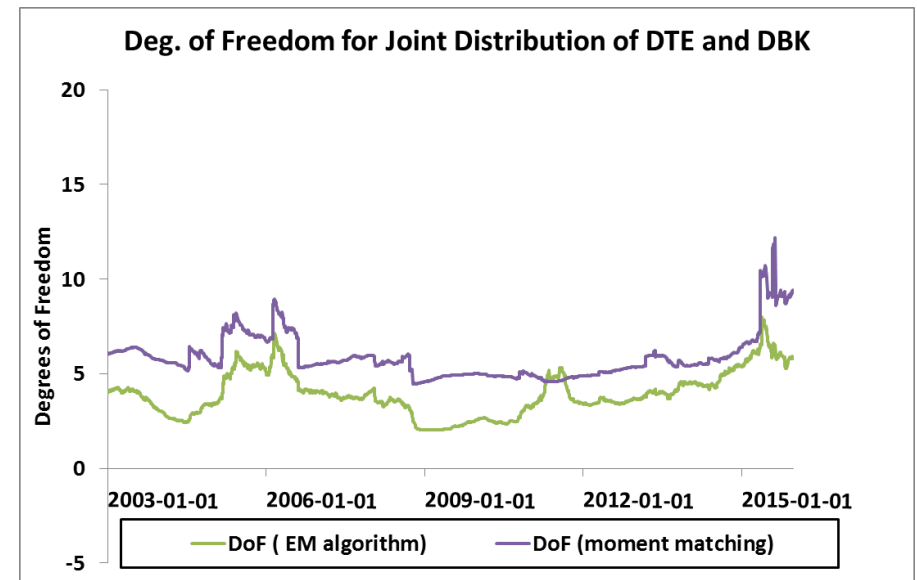
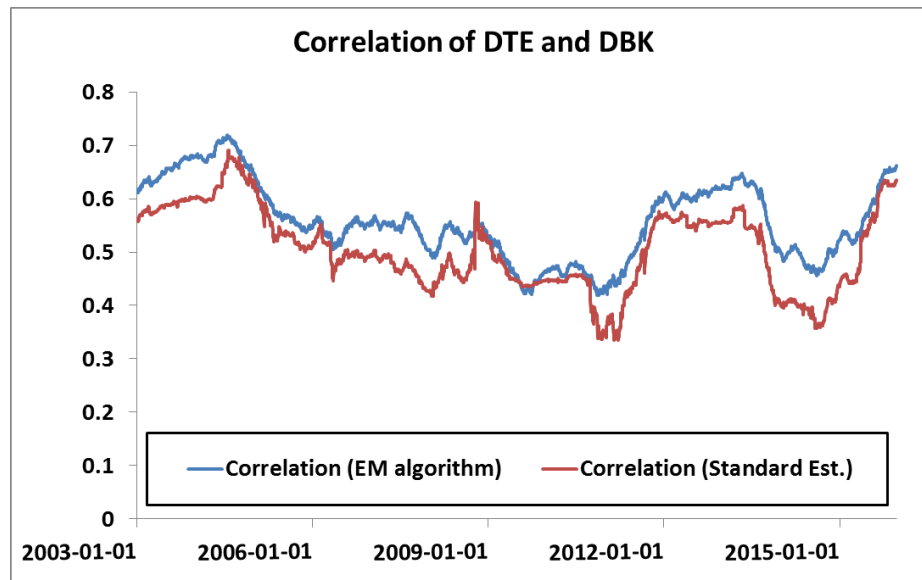
⁽¹⁾ C. Liu, D.B. Rubin, *ML Estimation of the t Distribution using EM and its Extensions*, ECM and ECME, Statistica Sinica 5 (1995), 19-39

⁽²⁾ X. Meng, D. van Dyk, *The EM Algorithm – an Old Folk-song Sung to a Fast New Tune*, J.R. Statist. Soc. B (1997), 59 (3), 511-567

⁽³⁾ W. Breyermann, D. Lüthi, *ghyp: A package on generalized hyperbolic distributions*, (2013)

Calibration example for multivariate Student – t- distribution

- » Example calibration for 2 return times series of DTE and DBK, using a 2 year rolling time window
- » Comparison of 2 methods (i) EM – algorithm (ii) Moment matching



- » The agreement between both approaches is fair at best, especially the degrees of freedom show strong deviations in certain periods
- » Also note that the moment-matching method assumes that the fourth moment exists

Practical experience: Calibrating the “degrees of freedom” of a Student-t distribution can be a challenge

Simulation

- » A random variable \mathbf{X} following a multivariate Student-t distribution with dispersion matrix Σ and k degrees of freedom can be represented as:

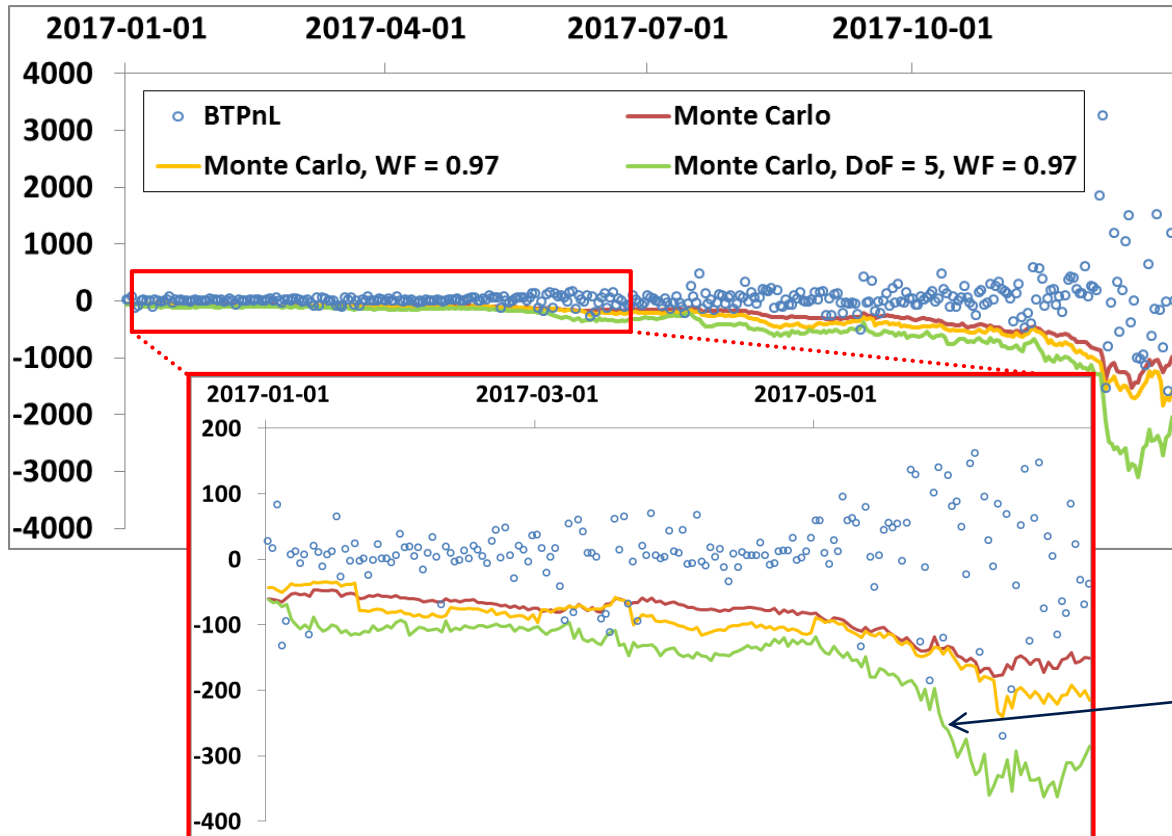
$$\mathbf{X} \sim \frac{\mathbf{Z}}{\sqrt{Y/k}}$$

where \mathbf{Z} is multivariate normally distributed with covariance matrix Σ and Y is a univariate $\chi^2(k)$ distributed random variable independent of \mathbf{Z} .

- » This relation can serve as basis for simulation:
 1. After calibration, simulate a multivariate normal distribution as explained before
 2. Independently, simulate a univariate $\chi^2(k)$ distributed random variable (for example using the fact that a $\chi^2(k)$ distributed random variable is the sum of k squared normal random variables)
 3. Use the relation above to generate multivariate Student-t distributed random variables

Use of fat-tailed distributions - example

» Example Portfolio 1 (BTC) in 2017:



pf1: Jan 2017 - Dec 2017			
Weighting Factor	#outlier	Basel	Kupiec
Monte Carlo (MC)	18	red	failed
MC , WF = 0.97	15	red	failed
MC, DoF = 5, WF = 0.97	5	green	passed

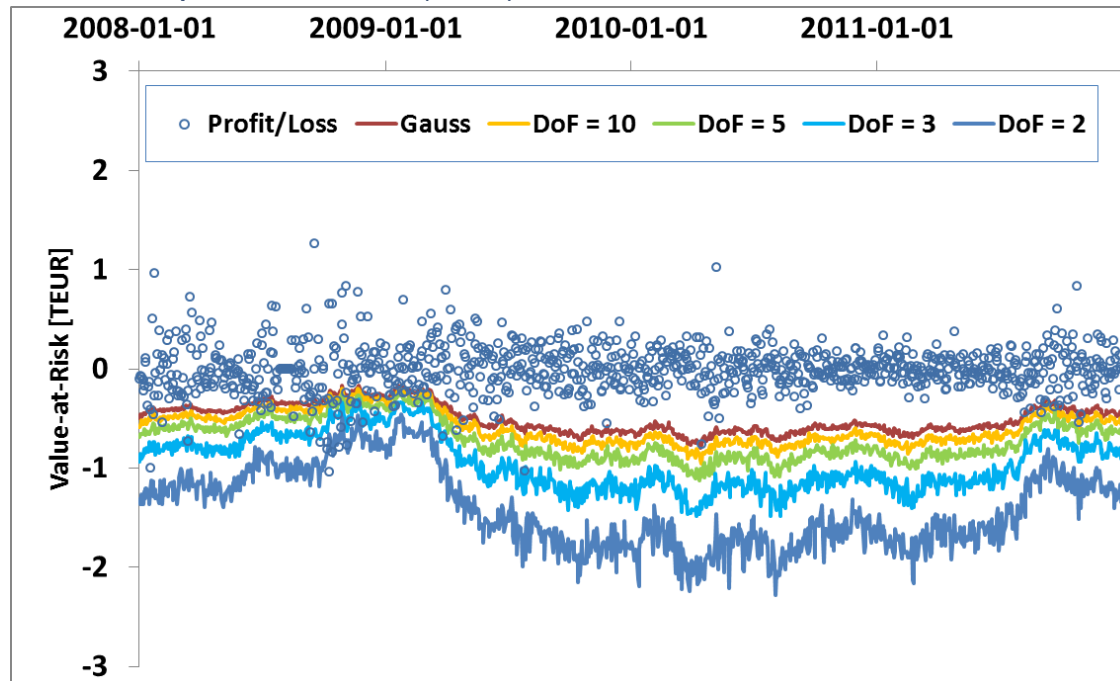
» Using a Student – t distribution with 5 degrees of freedom (DoF) improves the model sufficiently to pass the backtests

» This comes at the cost of making the model generally more conservative

Using a fat-tailed distribution can make the model significantly more conservative

Results for different degrees of freedom

» Example Portfolio 3 (DBK) from 2008 to 2011:



- » As expected the different degrees of freedom reflect roughly the differences between the 99% quantiles of the Student-t distribution
- » Decreasing degrees of freedom generally makes the risk figures more conservative, irrespective of market phase

» Jan 2008 – August 2009: Fewer outliers for decreasing number of degrees of freedom

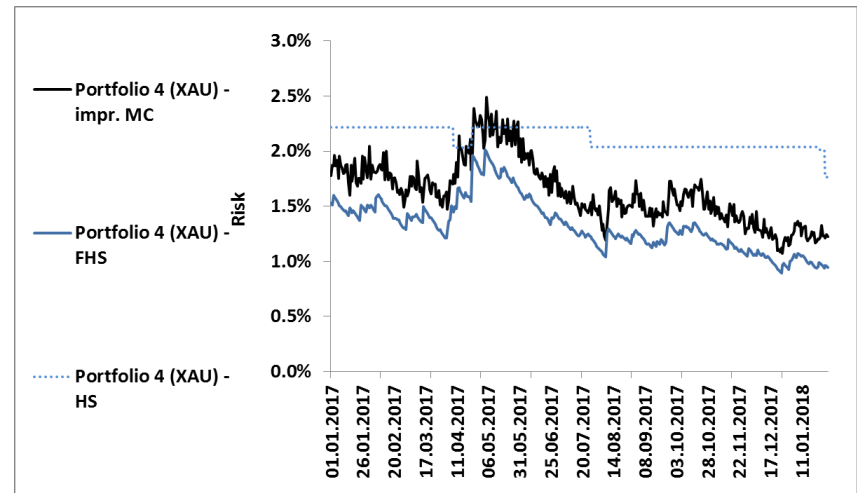
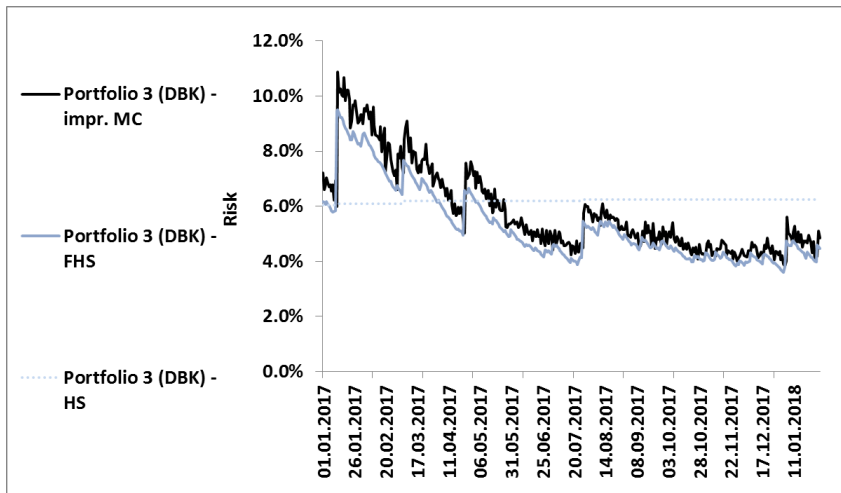
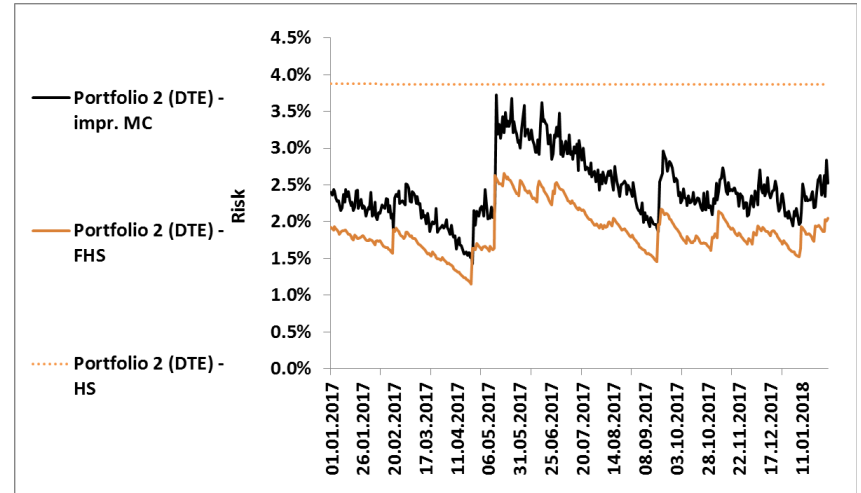
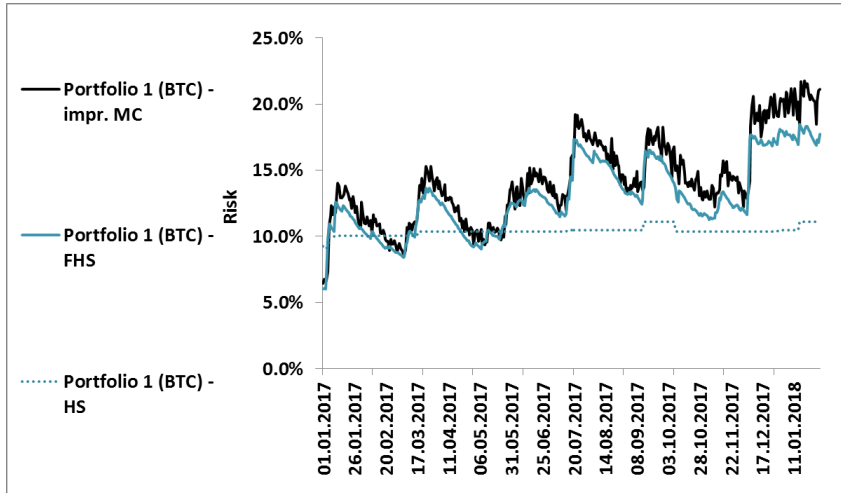
pf2: Jan 2008 - Aug 2009			
Weighting Factor	#outlier	result Basel	result Kupiec
Gauss	45	red	failed
k = 10	33	red	failed
k = 5	24	red	failed
k = 3	12	yellow	failed
k = 2	4	green	passed

» Sep 2009 – June 2011: Model very conservative

pf2: Sep 2009 - Jun 2011			
Weighting Factor	#outlier	result Basel	result Kupiec
Gauss	1	green	passed
k = 10	0	green	failed
k = 5	0	green	failed
k = 3	0	green	failed
k = 2	0	green	failed

Using a Student-t distribution can incorporate fat tails but can also easily become very conservative

Example portfolios – result overview (1/2)



















Example portfolios – result overview (2/2)

Date	31.01.2018	Value - at - Risk [TEUR]				Value-at-Risk / Market Value			
		Market Value [TEUR]	Hist. Sim.	Monte Carlo (Gauss)	Filtered Hist. Sim.	Monte Carlo (St.-5, WF = 0.97)	Hist. Sim.	Monte Carlo (Gauss)	Filtered Hist. Sim.
Portfolio 1 (BTC)	8.15	0.91	1.04	1.39	1.71	11.1%	12.8%	17.1%	21.0%
Portfolio 2 (DTE)	7.06	0.27	0.12	0.14	0.20	3.9%	1.7%	2.0%	2.8%
Portfolio 3 (DBK)	2.96	0.18	0.10	0.13	0.15	6.2%	3.3%	4.5%	5.1%
Portfolio 4 (XAU)	5.40	0.10	0.05	0.05	0.07	1.8%	1.0%	1.0%	1.2%
Portfolio 5	23.57	0.99	1.04	1.45	1.81	4.2%	4.4%	6.2%	7.7%

- » The standard Monte Carlo simulation has been improved by:
 - » Using a Student distribution with 5 degrees of freedom instead of a Gaussian distribution
 - » Calibrating the covariance matrix using a weighting factor of 0.97
- » Results are similar – but slightly more conservative - to those obtained by the filtered historical simulation with decay factor 0.97 and confirm the observations made earlier
- » Additional observations:
 - » For BTC the deviation between the Monte Carlo simulation and the filtered historical simulation is relatively small – this confirms again that BTC has a relatively fat-tailed return distribution

Combining weighted calibration and with a more realistic distributional assumption yields results similar to those obtained by filtered historical simulation

Filtered historical simulation & Monte Carlo simulation: comparison of approaches

	Filtered Historical Simulation	(Improved) Monte Carlo Simulation
Methodological soundness		
Reactivity / Accuracy	 	 
Complexity		
Transparency		
Dependency on data quality	 	
Calibration		 

Monte Carlo Simulation can be improved to reach a similar quality (and complexity) as filtered historical simulation

Summary and outlook

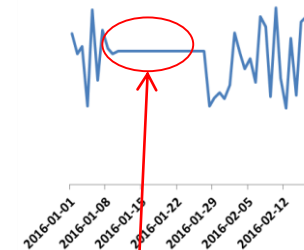
State – of – the art market risk models

- » We have looked at the two main modelling approaches for statistical market risk models as they have developed over the last years: **Filtered historical simulation** and **Monte Carlo simulation**
 - » General tendency: Methods become **more accurate** but also more **demanding on input data** (both quality and quantity)
 - » These models give a fairly good picture on the (market) risk of assets or portfolio of assets – **but only if the input data is of sufficient quality**

- » This is one of the drivers why certain topics become more and more important, also driven by regulators:
 - » **Model risk:** Risk introduced by wrong specification of model parameters
 - » **Data quality:** How to deal with **missing data**, **outliers** caused for example by technical problems, or **stale data**
 - » **Alternative (“non-statistical”) approaches:** Techniques like stress testing increasingly supplement statistical models

Date	Price 1	Price 2
2016-01-01	20.185	50.746
2016-01-02	20.080	50.348
2016-01-03	20.164	50.550
2016-01-04	NA	50.931
2016-01-05	NA	50.724
2016-01-06	20.150	50.235
2016-01-07	20.308	50.534
2016-01-08	20.389	50.440
2016-01-09	20.085	NA
2016-01-10	20.354	50.509
2016-01-11	20.049	50.419
2016-01-12	20.328	50.425

missing data



stale data

The past years have seen the development of sophisticated simulation approaches which bring along new challenges to be addressed in the future

New data sources, new models ?

- » New techniques (like the often - quoted “Machine Learning” - techniques) to extract information from all kinds of ‘alternative’ data sources have been developing in the last years
- » This may offer new approaches to explore in market risk measurement - some ideas:
 - › **Intraday risk:** Measure market sentiment using sources like Twitter

News › Business

Donald Trump wipes off \$1bn from Boeing's stock market value with single tweet

independent.co.uk, 16/12/2016

- › **Data quality:** Use new techniques to fill in missing or stale data (based for instance on news feeds or similar data sources) for which no price data is available

Thank you for your attention !



Dr Florian Baumann

Senior Manager

Tel +49 69 90737-448

Mobile +49 162 2630133

E-Mail Florian.Baumann@d-fine.de

d-fine

Berlin

Dusseldorf

Frankfurt

London

Munich

Vienna

Zurich

Headquarters

d-fine GmbH

An der Hauptwache 7

D-60313 Frankfurt/Main

Germany

Tel +49 69 90737-0

Fax +49 69 90737-200

www.d-fine.com

d-fine