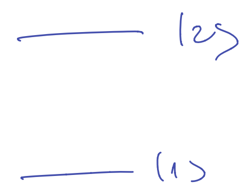
DAY 5: FRIDAY

reload:

(8) The quantized field



$$\hat H = \hat H_{at} + \hat H_{field} + \hat H_{int}$$

$$\hat H_{int} = -\mathbb{1}\,\hat{\vec d}\cdot\vec E\cdot\mathbb{1}$$

↑ now here quantized field,

$$\sim \sum_k (\dots a_k^\dagger + \dots a_k \dots)$$

$$\hat H_{int} = -\mathbb{1}\cdot\hat{\vec d}\cdot\hat{E}\cdot\mathbb{1} = \qquad \vec d_{12} = \vec d_{12}^{\,*} = \vec d$$

$$= -\left(\vec d_{12}\,|1\rangle\langle 2| + \vec d_{12}^{\,*}\,|2\rangle\langle 1|\right)\cdot\vec E =$$

$$= \hbar\sum_{\vec k}\left(g_{\vec k}\,\sigma_{12}\,a_k^\dagger + g_{\vec k}\,a_k\,\sigma_{21}\right)$$

Notations: $\sigma_{ij} = |i\rangle\langle j|$

$$g_{\vec k} = -\frac{\mathcal{E}_{\vec k}\,(\vec e_{\vec k}\cdot\vec d)}{\hbar}$$

What happened to the other 2 terms?

RWA → we want to remove the quickly oscillating terms / the energy non-conserving terms

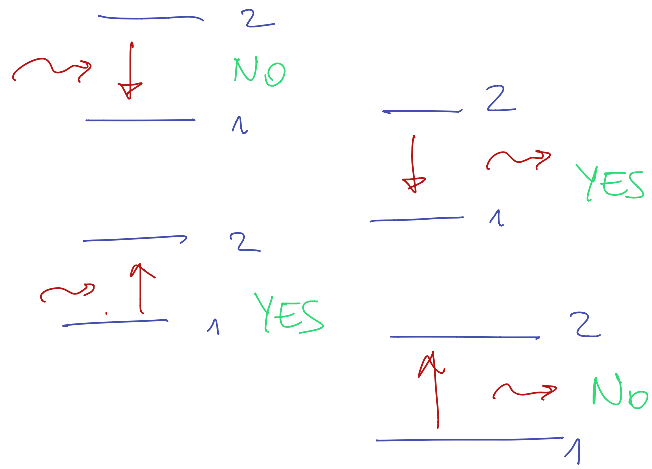
The 4 terms $\langle f| \quad |i\rangle$

$|1\rangle\langle 2|\, a$

$|1\rangle\langle 2|\, a^+$

$|2\rangle\langle 1|\, a$

$|2\rangle\langle 1|\, a^+$



We write the total Hamiltonian: (omitting constant terms!)

$$\boxed{H = \sum_{\vec k} \hbar\omega_{\vec k}\, a^+_{\vec k} a_{\vec k} + \frac{1}{2}\hbar\omega_{\vec k}\, \sigma_z + \hbar \sum_{\vec k} g_{\vec k} \left(\sigma_+ a_{\vec k} + \sigma_- a^+_{\vec k} \right)}$$

Jaynes – Cummings Hamiltonian

Notations:

$$\boxed{\begin{aligned} \sigma_z &= |2\rangle\langle 2| - |1\rangle\langle 1| \\ \sigma_+ &= |2\rangle\langle 1| \\ \sigma_- &= |1\rangle\langle 2| \end{aligned}}$$

reload from Thursday

We start with a simple calculation considering 1 field mode.

$$H = \underbrace{\hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega \sigma_z}_{H_0} + \underbrace{\hbar g(\sigma_+ a + \sigma_- a^\dagger)}_{H_{int}}$$

keep track now simultaneously of

- → atom state ($|1\rangle$ or $|2\rangle$)
- → # of photons

work in interaction picture

$$\hat{V} = \hbar g \left(\sigma_+ a e^{i\Delta t} + a^\dagger \sigma_- e^{-i\Delta t} \right)$$

$$\Delta = \omega_L - \omega_0$$

$$i\hbar \frac{\partial \psi}{\partial t} = V\psi$$

for a ψ of the form

photons

$$\psi(t) = \sum_n C_{1,n}(t)\,|1,n\rangle + C_{2,n}(t)\,|2,n\rangle$$

state of atom

according to energy conserv. we allow only

$$|2,n\rangle \to |1,n+1\rangle$$

coupled equations:

$$(*)\begin{cases} \dot{c}_{2,n} = -ig\sqrt{n+1}\, e^{i\Delta t}\, c_{1,n+1} \\ \dot{c}_{1,n+1} = -ig\sqrt{n+1}\, e^{-i\Delta t}\, c_{2,n} \end{cases}$$

Solve this!

$|c_{2,n}(t)|^2 \to$ probability to find at time t the atom in state $|2\rangle$ and n photons in the field.

$|c_{1,n}(t)|^2 \to$ - - - . state $|1\rangle$, n photons

Want to know just about field?

$$p(n) = |c_{2,n}(t)|^2 + |c_{1,n}(t)|^2$$

only population?

$$\sum_n \left(|c_{2,n}(t)|^2 + |c_{1,n}(t)|^2\right) = 1$$

Population inversion:

$$W(t) = \sum_n \left[\, |c_{2,n}(t)|^2 - |c_{1,n}(t)|^2 \right]$$

solving the equations (*) we obtain

$$W(t) = \sum_{n=0}^{\infty} \rho_{nn}(0) \left[\frac{\Delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2} \cos \Omega_n t \right]$$

$$\Omega_n^2 = \Delta^2 + 4g^2(n+1)$$

initial conditions:

$$\boxed{\begin{aligned} c_{2,n}(0) &= c_n(0) \\ c_{1,n+1}(0) &= 0 \end{aligned}}$$

$$\rho_{nn}(0) = |c_n(0)|^2$$

↪ probability for $\underline{n}$ photons at time $t=0$.

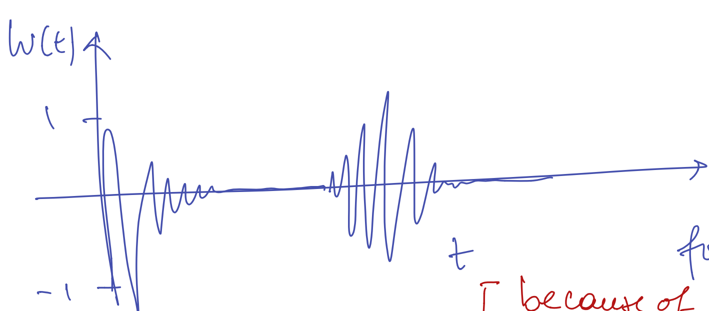
check this!

Even for $\rho_{nn}(0) = \delta_{n0}$ → only $n=0$ photons,

we still have

$$\boxed{W(t) = \frac{1}{\Delta^2 + 4g^2} \left[\Delta^2 + 4g^2 \cos\left(\sqrt{\Delta^2 + 4g^2}\, t\right) \right]}$$

Oscillations even in vacuum!



"collapse" and "revival" appear for $n \neq 1$

[because of superposition of cosine waves !!!]

This looks completely different than the Rabi oscillations in the semiclassical theory!!!

? Does this make sense?

A 2-level system, left alone, starts oscillating like crazy and experiences collapse & revival?

Even starting in vacuum from the excited state?

No, where is our spontaneous decay?

Solution to this dilemma:

the vacuum contains all field modes. A single mode is not the true story!

9. Weisskopf-Wigner theory of spontaneous emission

→ interaction Hamiltonian for many modes:

$$V = \hbar \sum_{\vec{k}} \left[g^*_{\vec{k}}(\vec{r}_0)\, \sigma_+ a_{\vec{k}}\, e^{i(\omega_0 - \omega_{\vec{k}})t} + h.c. \right]$$

$$g^*_{\vec{k}}(\vec{r}_0) = g_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{r}_0}$$

$\vec{r}_0$ → location of atom

Situation of interest for spontaneous decay:

$t=0$,
atom in excited state,
field in vacuum

$$\begin{cases} \overset{c_2}{\underset{}{\overset{|||}{c_{2,0}}}}(0) = 1 \\ c_{1,\vec{k}}(0) = 0 \end{cases}$$

wavefct.:

field

$$\psi(t) = c_2(t)\,|2,0\rangle + \sum_{\vec{k}} c_{1,\vec{k}}\,|1, 1_{\vec{k}}\rangle$$

↪ 1 photon in one of the modes!

atom

$$i\hbar \frac{\partial \psi}{\partial t} = V\psi$$

$$\Rightarrow \begin{cases} \dot{c}_2(t) = -i \sum_{\vec{k}} g^*_{\vec{k}}(\vec{r}_0)\, e^{i(\omega_0 - \omega_{\vec{k}})t}\, c_{1,\vec{k}}(t) \\ \dot{c}_{1,\vec{k}}(t) = -i g_{\vec{k}}(\vec{r}_0)\, e^{-i(\omega_0-\omega_{\vec{k}})t}\, c_2(t) \end{cases}$$

combine the two in an integro-diferential eq.

$$\boxed{\dot{c}_2(t) = -\sum_{\vec{k}} |g_{\vec{k}}(\vec{r}_0)|^2 \int_0^t dt'\, e^{i(\omega_0-\omega_k)(t-t')}\, c_2(t')}$$

this expression is exact.

Now some approximations.

1) go from $\sum_{\vec{k}}$ to $\int d^3\vec{k}$ → closely spaced modes

$$\sum_{\vec{k}} \rightarrow 2\frac{V}{(2\pi)^3}\int_{4\pi} d\Omega_{\vec{k}} \int_0^\infty dk\, k^2 \quad , k = \frac{\omega}{c}$$

$$g_{\vec{k}}(\vec{r}_0) = \frac{\omega_k}{2\hbar \varepsilon_0 V} d^2 \cos^2\theta \qquad (\theta \text{ is } \sphericalangle\ \vec{d}, \vec{e}_k)$$

$$\dot{c}_2(t) = -\frac{4d^2}{(2\pi)^2\, 6\hbar \varepsilon_0 c^3}\int_0^\infty d\omega_k\, \omega_k^3 \int_0^t dt'\, e^{i(\omega_0-\omega_k)(t-t')} c_2(t')$$

$$\omega_k \simeq \omega_0$$

$$\int_0^\infty \rightarrow \int_{-\infty}^\infty d\omega_k$$

So that we can use

$$\int_{-\infty}^{+\infty} d\omega_k\, e^{i(\omega_0-\omega_k)(t-t')} = 2\pi\,\delta(t-t')$$

$$\Rightarrow \boxed{\dot{c}_2(t) = -\frac{\Gamma}{2} c_2(t)} \quad \text{with} \quad \boxed{\Gamma = \frac{1}{4\pi\varepsilon_0}\frac{4\omega^3 d^2}{3\hbar c^3}}$$

$\Rightarrow$ $\varrho_{22} = |c_2(t)|^2 = e^{-\Gamma t}$ $\rightarrow$ exponential decay!!!

We have found our spontaneous decay!

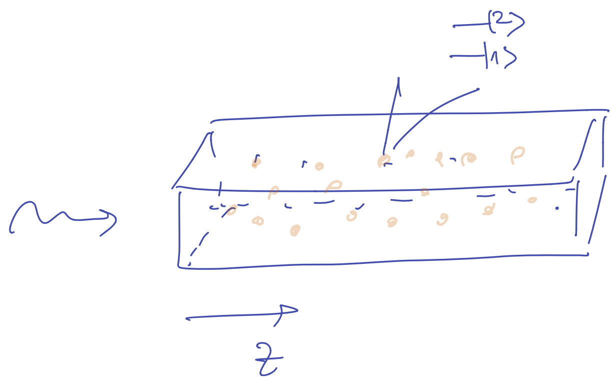
HERE : Applet Rabi oscillations
Rabi for loss of population in free system

10. Maxwell-Bloch equations

We turn back to the semi-classical theory.

Bloch equations: quantify the excitation in the atom.

Can we describe the pulse propagation?



a medium with many 2-level systems

laser field + response of the medium upstream propagate together.

$\rho(z,t)$

$\Omega_R(z,t) = \frac{1}{\hbar} \langle 2 | \vec{d} \cdot \vec{E} | 1 \rangle$ → space- and time-dependent

field creates polarization of the medium

$$P = \varepsilon_0 \chi E = N \langle d \rangle = N \, Tr \langle \rho d \rangle$$
$$= N \cdot \rho_{21} \cdot d$$

Polarization creates re-emission, new field.

wave equation:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2}$$

slowly-varying amplitudes → we approximate (neglect the second derivative

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{E}}{\partial z} = i\cdot\frac{2\pi}{\varepsilon_0 \lambda} N d \rho_{21}$$

for Ω

$$\frac{1}{c}\frac{\partial \Omega}{\partial t} + \frac{\partial \Omega}{\partial z} = i\eta\rho_{21}$$

$$\eta = \frac{N\sigma L \Gamma}{2L} = \frac{N\sigma\Gamma}{2} \rightarrow \text{optical thickness}$$

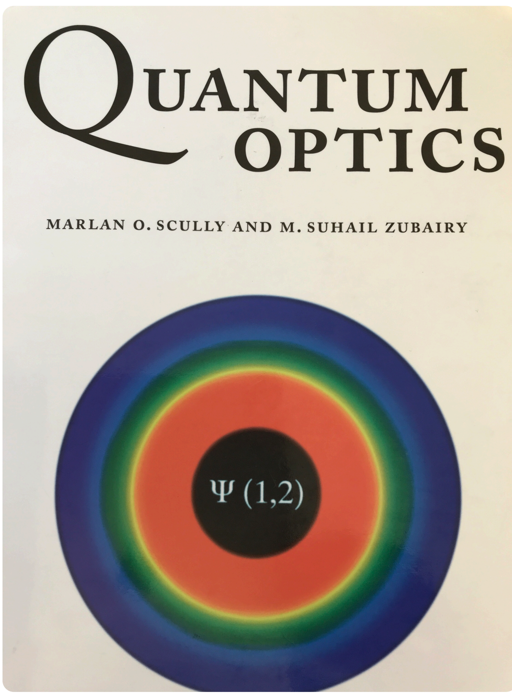
this is solved self-consistently together with the Bloch equations

$$\boxed{\begin{aligned}
\dot\rho_{22} &= -\gamma\rho_{22} + i\Omega(\rho_{12}-\rho_{21})\\
\dot\rho_{11} &= +\gamma\rho_{22} - i\Omega(\rho_{12}-\rho_{21})\\
\dot\rho_{12} &= -\left(i\Delta+\frac{\gamma}{2}\right)\rho_{12} - i\Omega(\rho_{11}-\rho_{22})\\
\frac{1}{c}\frac{\partial\Omega}{\partial t} &+ \frac{\partial\Omega}{\partial z} = i\eta\rho_{21}
\end{aligned}}$$

The Maxwell-Bloch eq.

Discussion during lecture: this condensed sketch of derivation is not at all convincing!!

Please check: Section 5.4 of "Quantum Optics" by M. Scully and M. Zubairy:



In addition, you can also check our derivation for the particular case of nuclear forward scattering (NFS) – see following slides – in New Journal of Physics

Please check also ⇒ New J. Phys 16, 013049 (2014)

Examples (slides)

- → superradiance
- → nuclear forward scattering
- → thick samples
- → thin samples
- → storage of photons
- → EIT
- → mimicking strong coupling
- → Summary slide (summary of lecture)?

THANK YOU FOR YOUR ATTENTION!!!