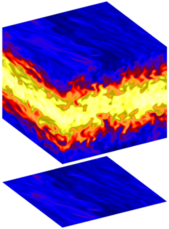
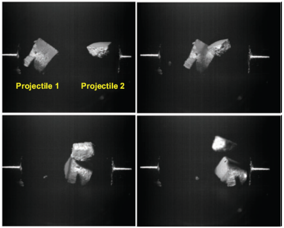
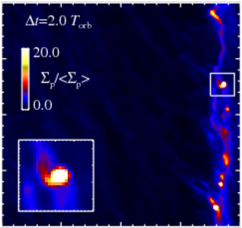
Lecture 3:
Growth of particles





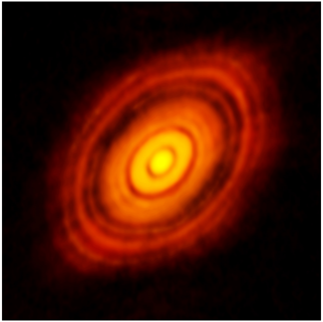


"Planet formation"

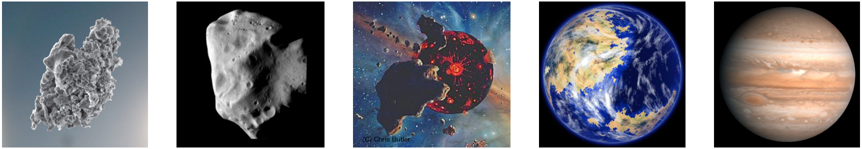
April 2016

Bertram Bitsch (Lund Observatory)

Conditions for planet formation



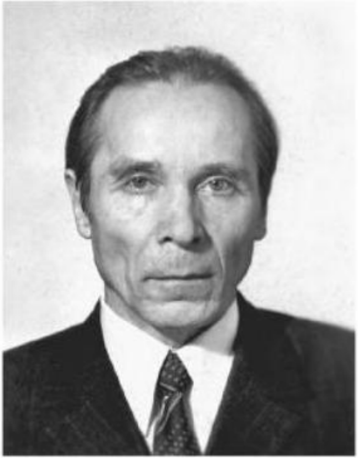
- Young stars are orbited by turbulent protoplanetary discs
- Disc masses of 10^{-4}–10^{-1} $M_\odot$
- Disc life-times of 1–10 million years



Planet formation paradigm

Planetesimal hypothesis:

Planets form in protoplanetary discs around young stars from dust and ice grains that stick together to form ever larger bodies

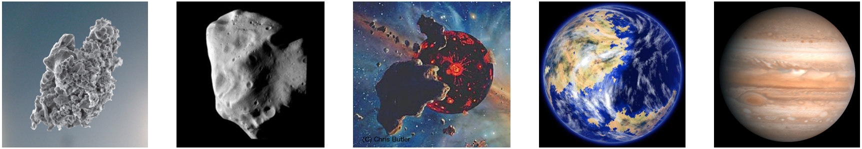


- Viktor Safronov (1917-1999):
  "father" of the planetesimal hypothesis

- "Evolution of the Protoplanetary Cloud and Formation of the Earth and the Planets" (1969, translated from Russian)

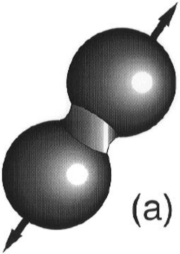
The four steps of planet formation

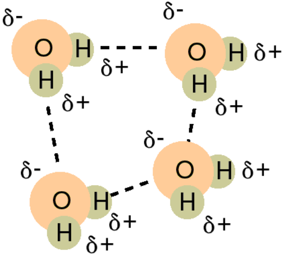
1. Dust to pebbles
   μm → dm: contact forces during collision lead to sticking
2. Pebbles to planetesimals
   dm → km: gravitational collapse of pebble clouds form planetesimals
3. Planetesimals to protoplanets
   km → 1,000 km: gravity (run-away accretion)
4. Protoplanets to planets
   Gas giants: $10\ M_\oplus$ core accretes gas ($< 10^7$ years)
   Terrestrial planets: protoplanets collide (10^7–10^8 years)



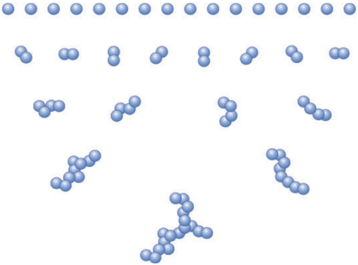
Sticking

- Colliding particle stick by the same forces that keep solids together (van der Waals forces such as dipole-dipole attraction)

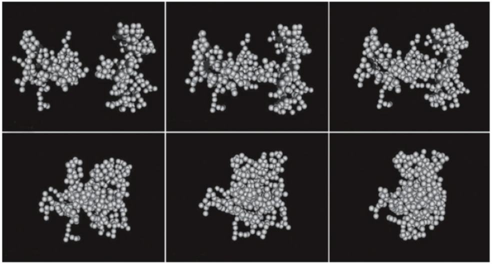




Dust experiments



(*Blum & Wurm*, 2008)

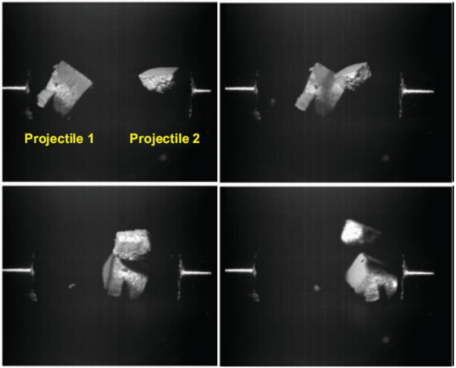


(*Paszun & Dominik*, 2006)

- Dust growth starts with μm-sized *monomers*
- Growth of dust aggregates by hit-and-stick
- Dust aggregates compactify in mutual collisions

Laboratory experiments

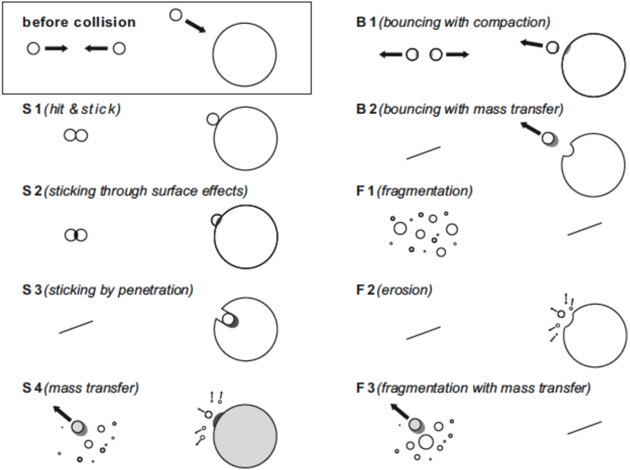
- Laboratory experiments used to probe sticking, bouncing and shattering of particles (labs e.g. in Braunschweig and Münster)
- Collisions between equal-sized macroscopic particles lead mostly to bouncing:



- From Blum & Wurm (2008)

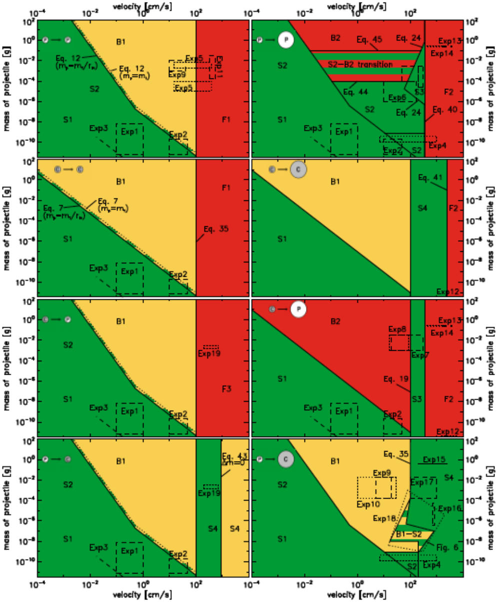
Collision regimes

- Güttler et al. (2010) compiled experimental results for collision outcomes with different particle sizes, porosities and speeds



Collision outcomes

- Güttler et al. (2010):
- Generally sticking or bouncing below 1 m/s and shattering above 1 m/s
- Sticking may be possible at higher speeds if a small impactor hits a large target



Drag force

Gas accelerates solid particles through drag force:

$$\frac{\partial \mathbf{v}}{\partial t} = \ldots - \frac{1}{\tau_\mathrm{f}}(\mathbf{v} - \mathbf{u})$$

Particle velocity

Gas velocity

In the Epstein drag force regime, when the particle is much smaller than the mean free path of the gas molecules, the friction time is

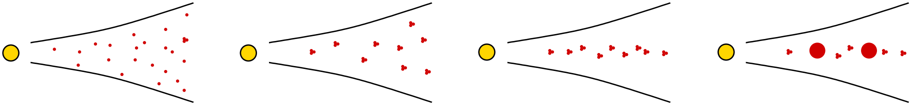
$$\tau_\mathrm{f} = \frac{a_\bullet \rho_\bullet}{c_\mathrm{s} \rho_\mathrm{g}}$$

$a_\bullet$: Particle radius
$\rho_\bullet$: Material density
c_s: Sound speed
ρ_g: Gas density

Important nondimensional parameter in protoplanetary discs:

$\Omega_\mathrm{K} \tau_\mathrm{f}$ (*Stokes number*)

Sedimentation

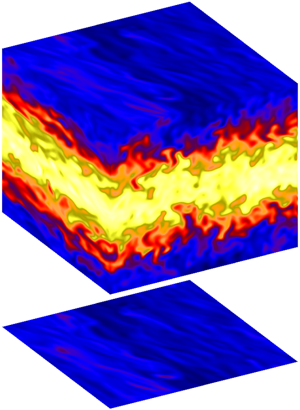


- Dust grains coagulate and gradually decouple from the gas
- Sediment to form a thin mid-plane layer in the disc
- Planetesimals form by continued coagulation or self-gravity (or combination) in dense mid-plane layer
- Turbulent diffusion prevents the formation of a very thin mid-plane layer

Diffusion-sedimentation equilibrium

Diffusion-sedimentation equilibrium:

$$\frac{H_{\rm dust}}{H_{\rm gas}} = \sqrt{\frac{\delta_{\rm t}}{\Omega_{\rm K}\tau_{\rm f}}}$$



$H_{\rm dust}$ = scale height of dust layer

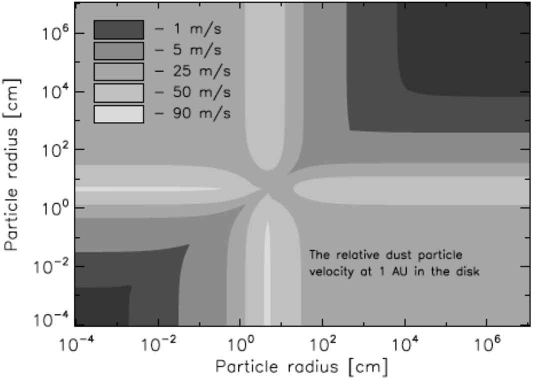
$H_{\rm gas}$ = scale height of gas

$\delta_{\rm t}$ = turbulent diffusion coefficient, like α-value ($D = \delta H c_s$)

$\Omega_{\rm K}\tau_{\rm f}$ = Stokes number, proportional to radius of solid particles

Turbulent collision speeds

- Turbulent gas accelerates particles to high collision speeds:



(Brauer et al. 2008; based on Weidenschilling & Cuzzi 1993)

⇒ Small particles follow the same turbulent eddies and collide at low speeds

⇒ Larger particles collide at higher speeds because they have different trajectories

Terminal velocity approximation

- Equation of motion of particles ($\mathbf{v}$) and gas ($\mathbf{u}$)

$$\begin{aligned}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= -\boldsymbol{\nabla}\Phi - \frac{1}{\tau_\mathrm{f}}(\mathbf{v}-\mathbf{u}) \\ \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} &= -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}P\end{aligned}$$

- Particles do not care about the gas pressure gradient since they are very dense
- Subtract the two equations from each other and look for equilibrium

$$\frac{\mathrm{d}(\mathbf{v}-\mathbf{u})}{\mathrm{d}t} = -\frac{1}{\tau_\mathrm{f}}(\mathbf{v}-\mathbf{u}) + \frac{1}{\rho}\boldsymbol{\nabla}P = 0$$

- In equilibrium between drag force and pressure gradient force the particles have their *terminal velocity* relative to the gas

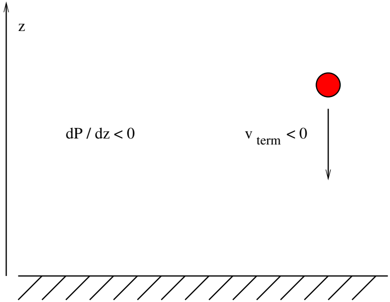
$$\delta\mathbf{v} = \tau_\mathrm{f}\frac{1}{\rho}\boldsymbol{\nabla}P$$

⇒ **Particles move towards the direction of higher pressure**

Ball falling in Earth's atmosphere

$$\mathbf{v}_{\rm term} = \tau_{\rm f} \frac{1}{\rho} \boldsymbol{\nabla} P$$

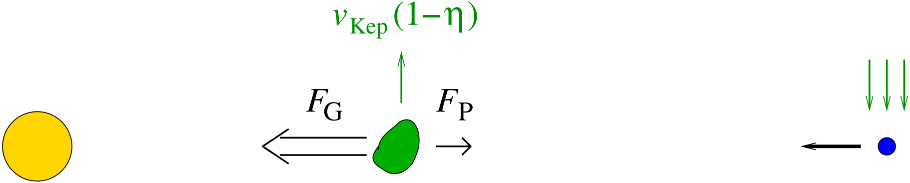
- Ball falling in Earth's atmosphere:



- Pressure is falling with height, so $\mathrm{d}P/\mathrm{d}z < 0$ and thus $v_{\rm term} < 0$

⇒ *Ball is seeking the point of highest pressure*

Radial drift



- Disc is hotter and denser close to the star
- Radial pressure gradient force mimics decreased gravity ⇒ gas orbits slower than Keplerian
- Particles do not feel the pressure gradient force and want to orbit Keplerian
- Headwind from sub-Keplerian gas drains angular momentum from particles, so they spiral in through the disc
- Particles sublimate when reaching higher temperatures close to the star

Sub-Keplerian motion

- Balance between gravity, centrifugal force and pressure gradient force:

$$0 = -\frac{GM_\star}{r^2} + \Omega^2 r - \frac{1}{\rho}\frac{\partial P}{\partial r}$$

- Δv is the velocity difference between gas and dust

$$\Delta v = -\frac{1}{2}\left(\frac{H}{r}\right)^2 \frac{\partial \ln P}{\partial \ln r} v_\mathrm{K} \equiv -\eta v_\mathrm{K}$$

- Use $H/r = (c_\mathrm{s}/\Omega_\mathrm{K})/(v_\mathrm{K}/\Omega_\mathrm{K}) = c_\mathrm{s}/v_\mathrm{K}$ to obtain the final expression

$$\Delta v = -\frac{1}{2}\frac{H}{r}\frac{\partial \ln P}{\partial \ln r} c_\mathrm{s}$$

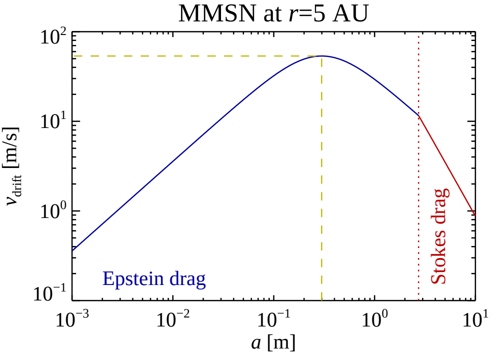
- Particles do not feel the global pressure gradient and want to orbit Keplerian ⇒ headwind from the sub-Keplerian gas

Radial drift

Balance between drag force and head wind gives radial drift speed
(Adachi et al. 1976; Weidenschilling 1977)

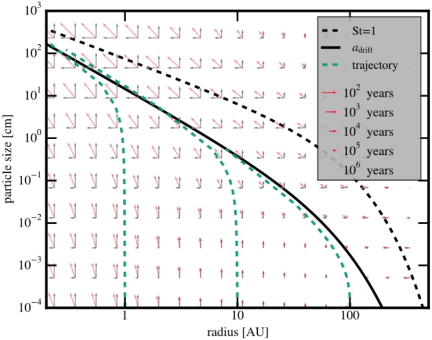
$$v_{\mathrm{drift}} = -\frac{2\Delta v}{\Omega_{\mathrm{K}}\tau_{\mathrm{f}} + (\Omega_{\mathrm{K}}\tau_{\mathrm{f}})^{-1}}$$

for Epstein drag law $\tau_{\mathrm{f}} = a\rho_{\bullet}/(c_{\mathrm{s}}\rho_{\mathrm{g}})$

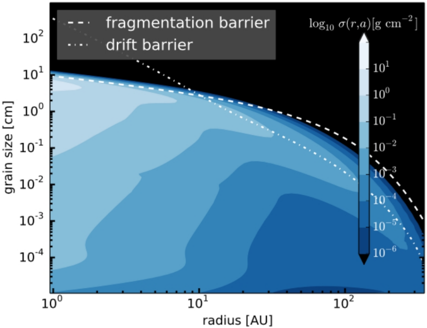


- MMSN $\Delta v \sim 50 \ldots 100$ m/s
- Drift time-scale of 100 years for particles of 30 cm in radius at 5 AU

Drift-limited growth



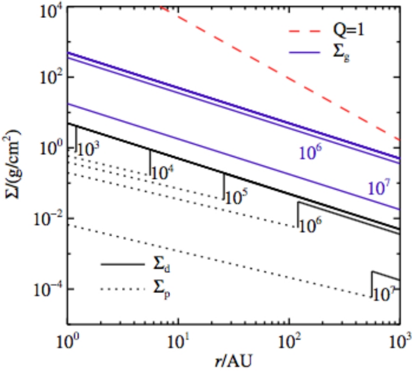
(*Birnstiel et al.*, 2015)

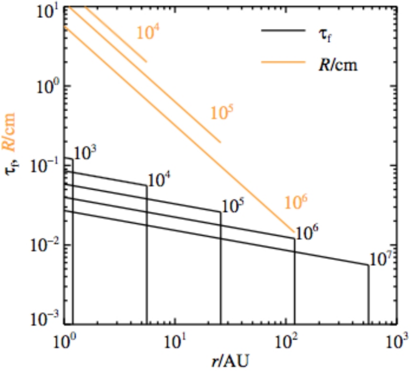


(*Testi et al.*, 2014)

- Particles in the outer disc grow to a characteristic size where the growth time-scale equals the radial drift time-scale (*Birnstiel et al. 2012*)
- Growth time-scale $t_{\rm gr} = R/\dot{R}$, drift time-scale $t_{\rm dr} = r/\dot{r}$
- Yields dominant particle Stokes number $\mathrm{St} \approx \frac{\sqrt{3}}{8} \frac{\epsilon_{\rm p}}{\eta} \frac{\Sigma_{\rm p}}{\Sigma_g}$, with $\epsilon \sim 1$ the sticking efficiency (*Lambrechts & Johansen, 2014*)
- Here the pebble column density can be obtained from the pebble mass flux through $\dot{M}_{\rm p} = 2\pi v_{\rm r} \Sigma_{\rm p}$

Radial pebble flux





- The pebble mass flux can be calculated from the pebble formation front that moves outwards with time (*Lambrechts & Johansen, 2014*)
- The final Stokes number is ∼ 0.1 inside 10 AU and ∼ 0.02 outside of 10 AU
- The drift-limited solution shows a fundamental limitation to particle growth
- Inclusion of bouncing and fragmentation results in even smaller particle sizes

Coagulation and radial drift

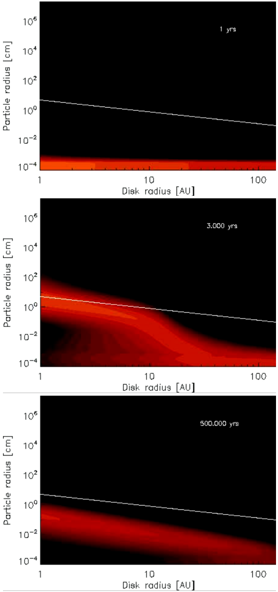
- Coagulation equation of dust particles can be solved by numerical integration

- We start with μm-sized particles and let the size distribution evolve by sticking and fragmentation

- The head wind from the gas causes cm particles to spiral in towards the star

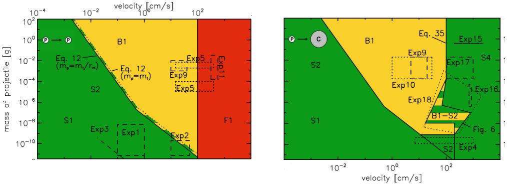
$\Rightarrow$ All solid material lost to the star within a few million years (*radial drift barrier*)

- Inclusion of particle fragmentation worsens the problem in the inner disc (*fragmentation barrier*)

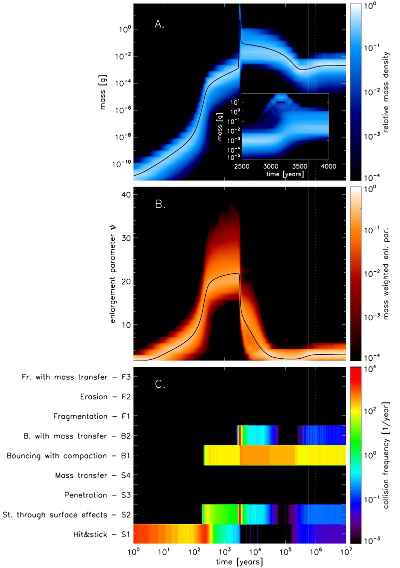


(Brauer et al. 2008)

Bouncing barrier

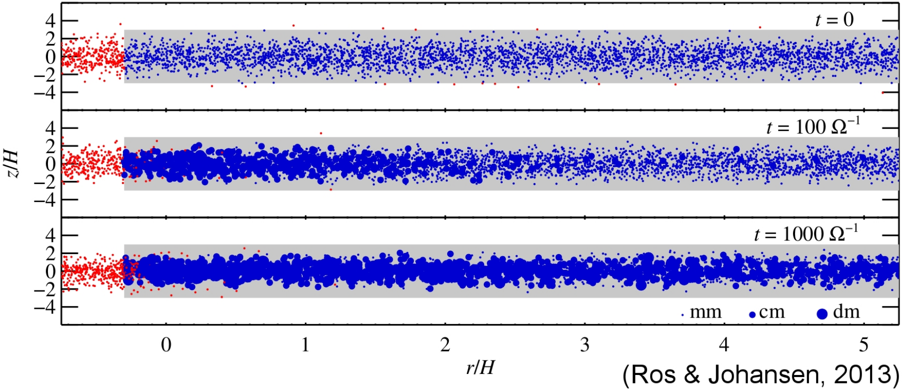


(*Zsom et al.*, 2010)



- Collisions between dust aggregates can lead to sticking, bouncing or fragmentation (*Güttler et al.*, 2010)
- Sticking for low collision speeds and small aggregates
- Bouncing prevents growth beyond mm sizes (*bouncing barrier*)
- Further growth may be possible by mass transfer in high-speed collisions (*Windmark et al.*, 2012) or by ice condensation (*Ros & Johansen*, 2013), but stops at radial drift barrier

Growth at ice lines



(Ros & Johansen, 2013)

- The radial ice-line feeds vapour directly into the mid-plane

⇒ Growth to dm-sized ice balls

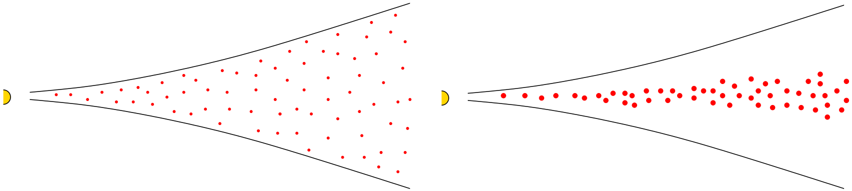
⇒ Turbulent diffusion mixes growing pebbles in the entire cold region

⇒ Future models of coagulation *and* condensation could yield large enough particle sizes for streaming instabilities to become important

Planetesimal formation by coagulation

- Coagulation works well to form cm-sized particles
- Radial drift, shattering, and bouncing prevent further growth
- Either there is something we do not understand about coagulation (sticky organical compounds e.g.) . . .
- . . . or we are missing some important piece of physics (maybe filling factor plays a role? (*Kataoka et al. 2013*))

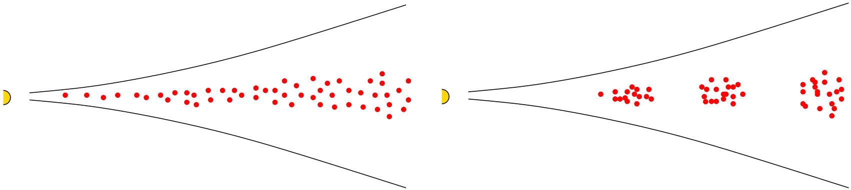
Planetesimal formation by gravitational instability



- Dust and ice particles in a protoplanetary disc coagulate to cm-sized pebbles and rocks
- Pebbles and rocks *sediment* to the mid-plane of the disc
- Further growth frustrated by high-speed collisions ($>$1–10 m/s) which lead to erosion and bouncing *(Blum & Wurm 2008)*
- Layer *not* dense enough for gravitational instability

⇒ **Need some way for particle layer to get dense enough to initiate gravitational collapse**

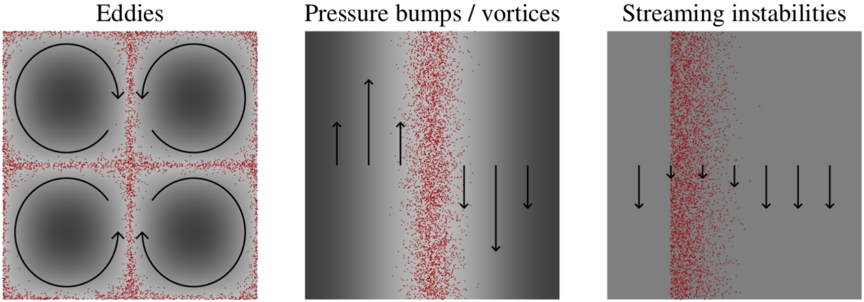
How turbulence aids planetesimal formation



1. *Passive concentration* as particles pile up in long-lived pressure bumps and vortices excited in the turbulent gas flow
(Barge & Sommeria 1995; Klahr & Bodenheimer 2003; Johansen et al. 2007)

2. *Active concentration* as particles make dense filaments and clumps to protect themselves from gas friction
(Youdin & Goodman 2005; Johansen & Youdin 2007; Johansen et al. 2009; Bai & Stone 2010a,b,c)

Particle concentrations



Eddies

Pressure bumps / vortices

Streaming instabilities

$l \sim \eta \sim 1$ km, St $\sim 10^{-5}$–10^{-4}

$l \sim 1$–$10\,H$, St ~ 0.1–10

$l \sim 0.1\,H$, St ~ 0.01–1

Three ways to concentrate particles: (Johansen et al., 2014, arXiv:1402.1344)

- Between small-scale low-pressure eddies
  (Squires & Eaton, 1991; Fessler et al., 1994; Cuzzi et al., 2001, 2008; Pan et al., 2011)
- In pressure bumps and vortices
  (Whipple, 1972; Barge & Sommeria, 1995; Klahr & Bodenheimer, 2003; Johansen et al., 2009a)
- By streaming instabilities
  (Youdin & Goodman, 2005; Johansen & Youdin, 2007; Johansen et al., 2009b; Bai & Stone, 2010a,b,c)

Roche density

- Protoplanetary discs are gravitationally unstable if the parameter Q is smaller than unity *(Safronov 1960; Toomre 1964)*

$$Q = \frac{c_{\mathrm{s}}\Omega}{\pi G \Sigma} < 1$$

- The column density can be written in terms of the scale height and the mid-plane density

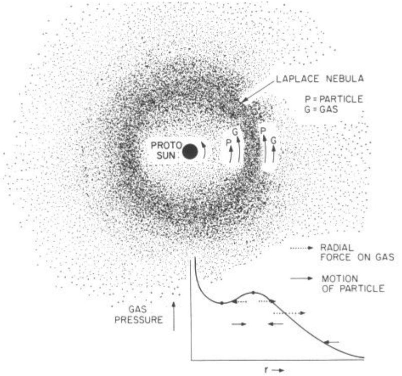
$$\Sigma \approx H\rho_0$$

- Turn the gravitational instability criterion into a criterion for the density

$$\rho_0 > \rho_{\mathrm{R}} \approx \frac{\Omega^2}{G} \approx \frac{M_\star}{r^3}$$

- The Roche density is $\rho_{\mathrm{R}} \approx 6 \times 10^{-7}\ \mathrm{g/cm^3}$ at 1 AU, the mid-plane gas density is $\rho_0 \approx 1.4 \times 10^{-9}\ \mathrm{g/cm^3}$

Pressure bumps



EFFECT OF GAS PRESSURE GRADIENT ON PARTICLE MOTION

Fig. 1.

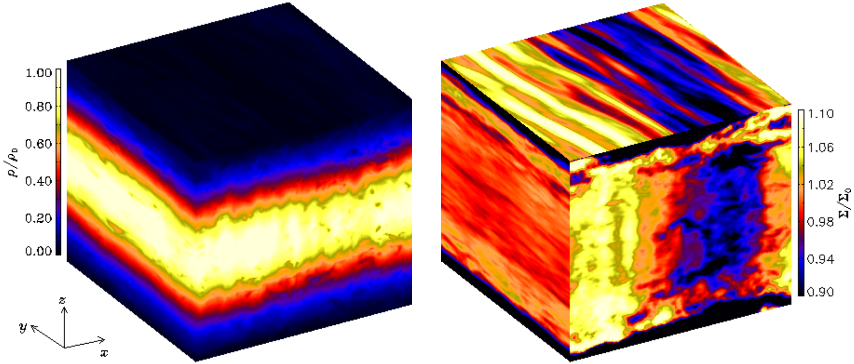
(Figure from Whipple 1972)

- Particles seek the point of highest pressure

⇒ Particles get trapped in *pressure bumps*

- Achieve high enough *local* density for gravitational instability and planetesimal formation

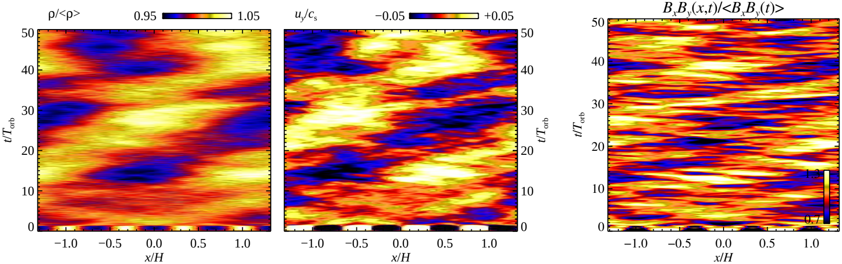
High-pressure regions



(Johansen, Youdin, & Klahr 2009)

- Gas density shows the expected vertical stratification
- Gas column density shows presence of large-scale pressure fluctuations with variation only in the radial direction
- Pressure fluctuations of order 10%

Stress variation and pressure bumps



- Mass accretion rate and column density:

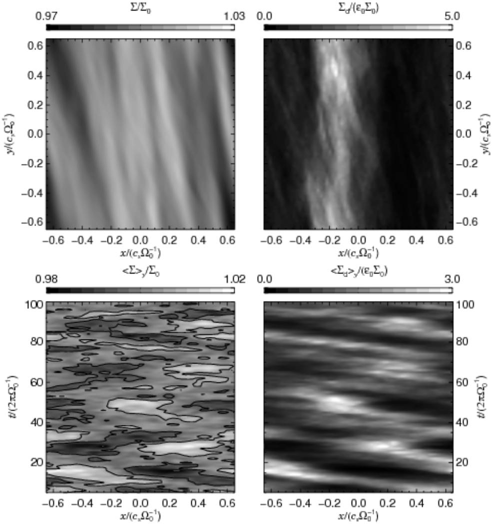
$$\dot{M} = 3\pi\Sigma\nu_{\rm t} \quad \Rightarrow \quad \Sigma = \frac{\dot{M}}{3\pi\nu_{\rm t}}$$

$$\nu_{\rm t} = \alpha c_{\rm s} H$$

⇒ Constant $\dot{M}$ and constant α yield $\Sigma \propto r^{-1}$

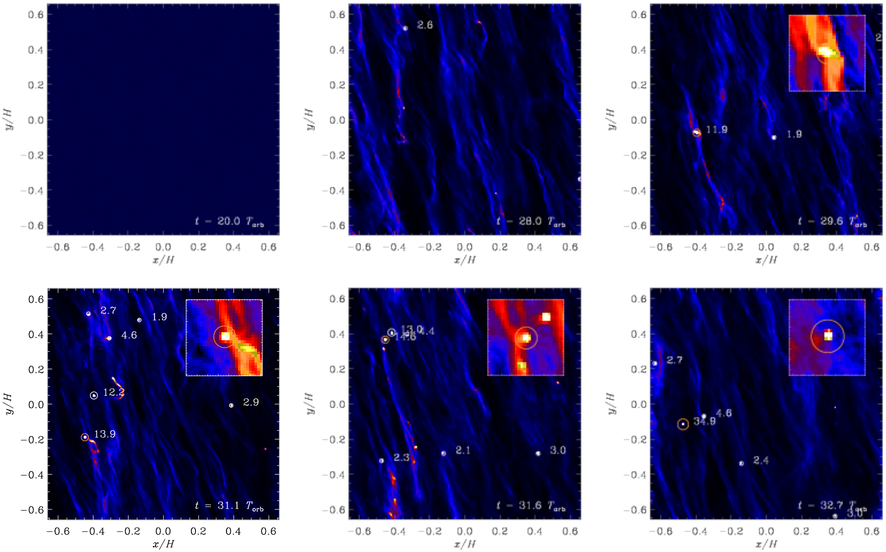
⇒ Radial variation in α gives pressure bumps

Particle trapping



- Strong correlation between high gas density and high particle density
  (Johansen, Klahr, & Henning 2006)

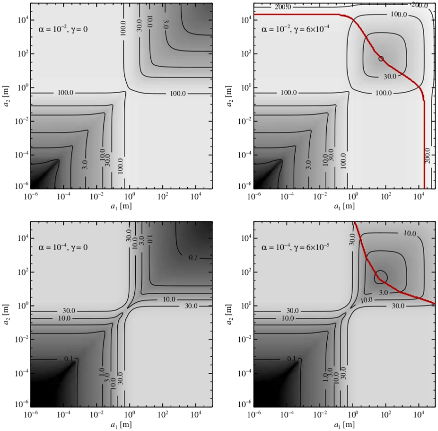
Forming planetesimals in pressure bumps



(Johansen et al. 2011)

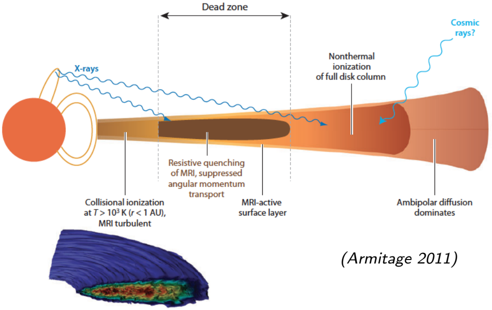
The double-edged sword called turbulence

- ☺ Turbulence can excite long-lived pressure bumps which trap particles
- ☹ Turbulence excites high relative particle speeds between particles as well as between planetesimals

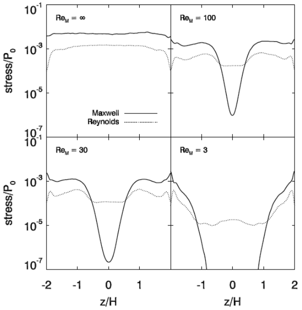


(*Johansen et al. 2014*)

Dead zone and layered accretion



(Armitage 2011)



(Gammie 1996, Fleming & Stone 2003, Oishi et al. 2007)

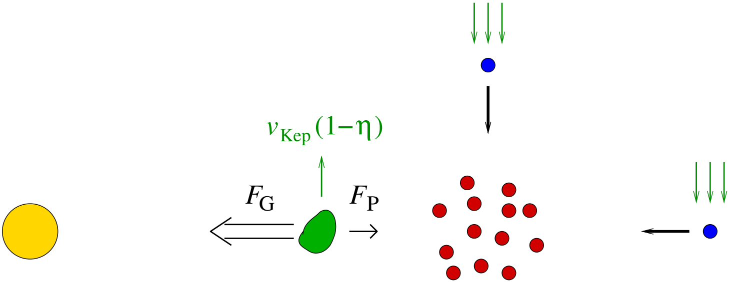
- Cosmic rays do not penetrate to the mid-plane of the disc, so the ionisation fraction in the mid-plane is too low to sustain MRI

⇒ Accretion in active surface layers

⇒ Weak turbulence and low collision speeds in the *dead zone*

Streaming instability

- Gas orbits slightly slower than Keplerian
- Particles lose angular momentum due to headwind
- Particle clumps locally reduce headwind and are fed by isolated particles

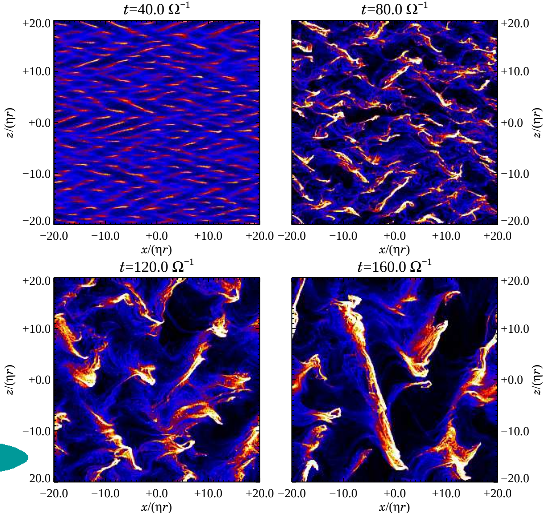


⇒ Youdin & Goodman (2005): **“Streaming instability”**

- *Shear instabilities* such as Kelvin-Helmholtz instability and magnetorotational instability feed on spatial variation in the gas velocity
- *Streaming instabilities* feed on velocity difference between two components (gas and particles) at the same location

Clumping

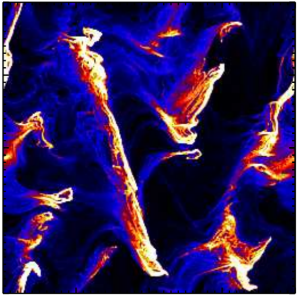
Linear and non-linear evolution of radial drift flow of meter-sized boulders:



⇒ Strong clumping in non-linear state of the streaming instability

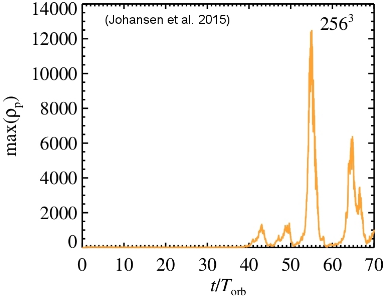
(Youdin & Johansen 2007, Johansen & Youdin 2007)

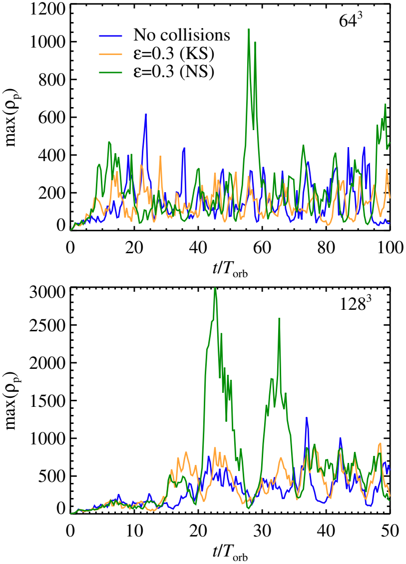
Why clump?



Particle density

- Particle density up to 3000 times local gas density
- Criterion for gravitational collapse: $\rho_\mathrm{p} \gtrsim \Omega^2/G \sim 100\rho_\mathrm{g}$
- Maximum density increases with increasing resolution


(Johansen et al. 2015)



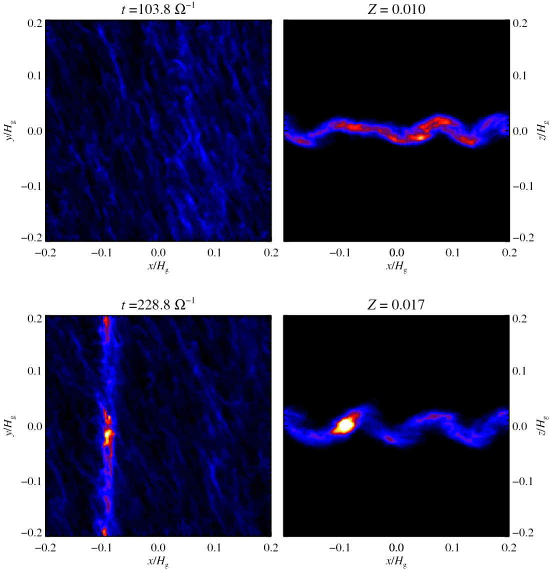
Sedimentation of 10 cm rocks

- Streaming instability relies on the ability of solid particles to accelerate the gas towards the Keplerian speed

⇒ Efficiency increases with the metallicity of the gas

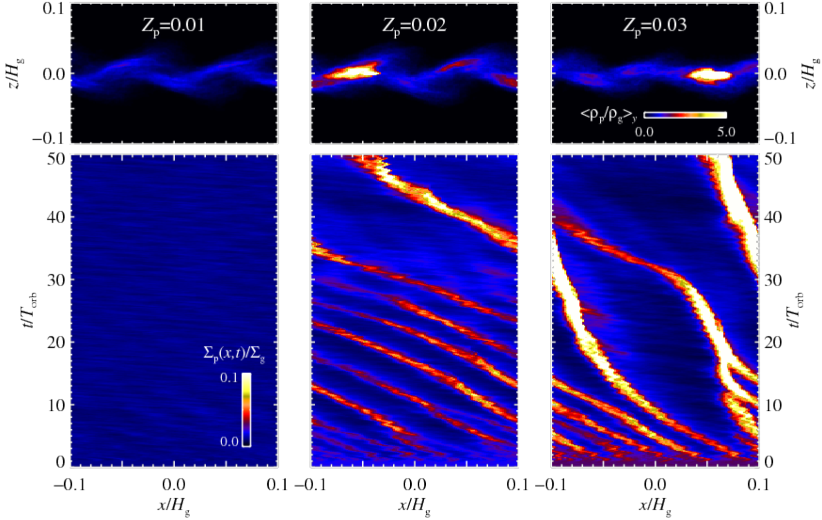
- Solar metallicity: turbulence caused by the streaming instability puffs up the mid-plane layer, but no clumping

- Dense filaments form spontaneously above $Z \approx 0.015$



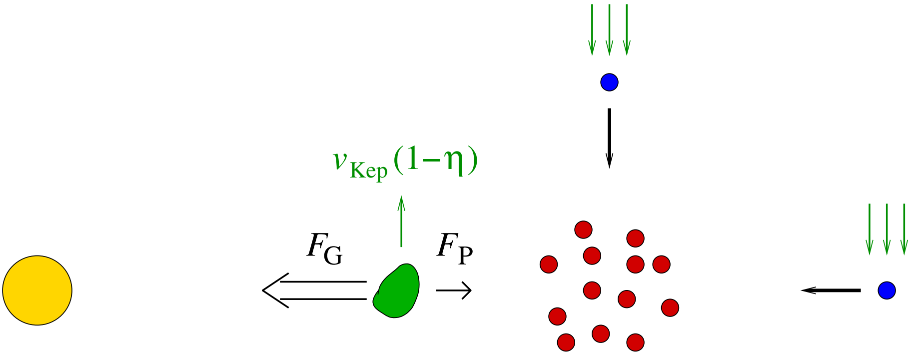
Dependence on metallicity

- Particles sizes 3–12 cm at 5 AU, 1–4 cm at 10 AU
- Increase pebble abundance $\Sigma_{\rm par}/\Sigma_{\rm gas}$ from 0.01 to 0.03



Why is metallicity important?

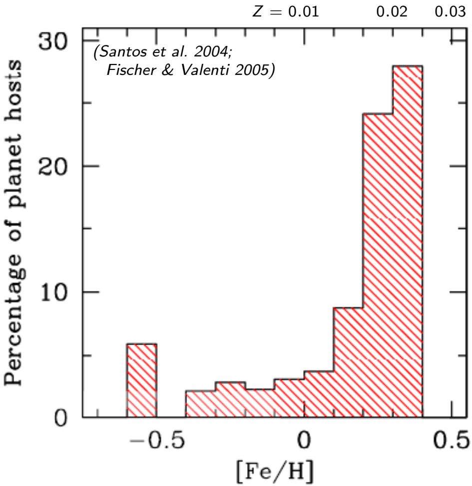
- Gas orbits slightly slower than Keplerian
- Particles lose angular momentum due to headwind
- Particle clumps locally reduce headwind and are fed by isolated particles



- *Clumping relies on particles being able to accelerate the gas towards Keplerian speed*

Metallicity of host star

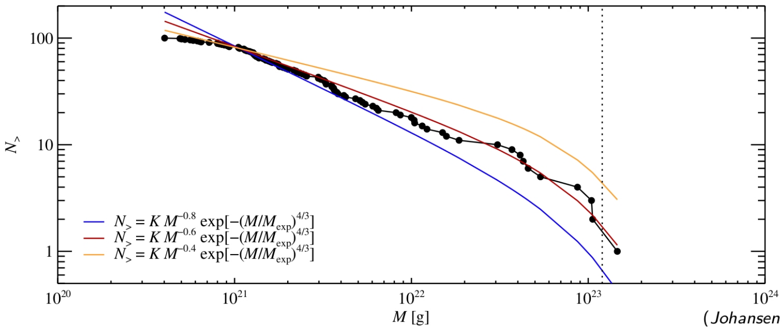
- First planet around solar-type star discovered in 1995 *(Mayor & Queloz 1995)*
- Today several thousand exoplanets known
- Exoplanet probability increases sharply with metallicity of host star



⇒ Expected due to efficiency of core accretion and pebble accretion
(Ida & Lin 2004; Mordasini et al. 2009; Lambrechts & Johansen 2014)

⇒ . . . but planetesimal formation may play equally big part
(Johansen et al. 2009)

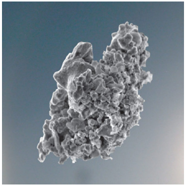
Planetesimal birth sizes

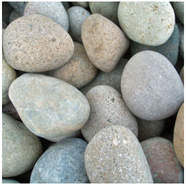


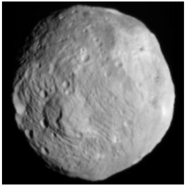
(*Johansen et al.*, 2015)

- Cumulative size distribution is less affected by noise than the differential size distribution
- Well-fitted by an exponentially tapered power law
- Most of the mass resides around the knee
- Small planetesimals dominate in number
- Can be compared to the asteroid belt: largest planetesimal has Ceres size

The “clumping scenario” for planetesimal formation

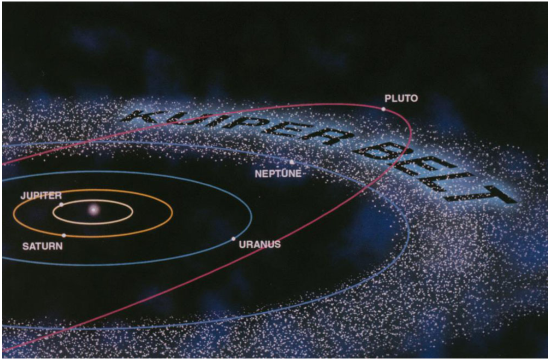






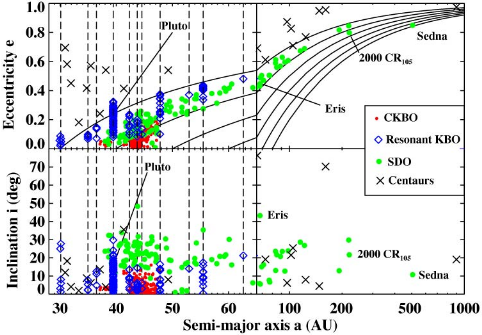
1. Dust growth by coagulation to a few cm
2. Spontaneous clumping through streaming instabilities and in pressure bumps
3. Gravitational collapse to form 100–1000 km radius planetesimals

Trans-Neptunian objects



- The orbits of trans-Neptunian objects (TNOs) lie entirely or in part beyond the orbit of Neptune
- TNOs constitute the overwhelming majority of minor bodies in the solar system
- There are 26 asteroids larger than 100 km in radius – the corresponding number of large objects in the Kuiper belt is closer to 5,000
- Divided into centaurs, scattered disc objects, classical Kuiper belt objects, and Oort cloud objects

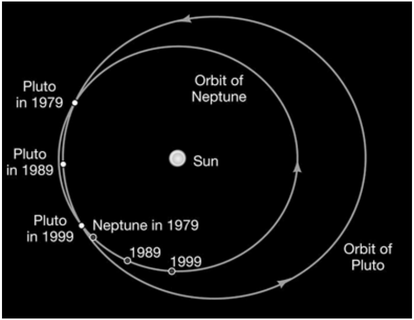
Classification of trans-Neptunian objects

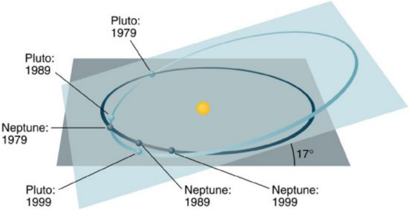


(Chiang et al. 2007)

- Kuiper belt objects reside beyond the orbit of Neptune
- Pluto trapped in 3:2 resonance with Neptune - result of outwards migration of Neptune ⇒ Nice model (lecture 5)
- Scattered disc objects have high *e* and perihelion distance between 33 and 40 AU
- Centaurs have perihelion within 30 AU - source of Jupiter family comets
- Classical KBOs have low *e* and semimajor axes between 37 and 48 AU - future target of New Horizons

Pluto's orbit





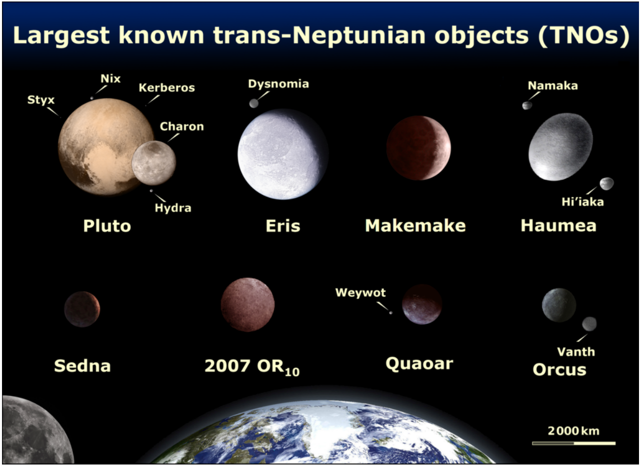
- Pluto's orbit is quite eccentric and crosses the orbit of Neptune
- Pluto avoids close encounters with Neptune because
  - Pluto is in a 3:2 resonance with Neptune so that Neptune is approximately 45 degrees behind or ahead of Pluto at Pluto's perihelion
  - Pluto's orbit is inclined relative to Neptune's, so Pluto is actually below Neptune where their projected orbits overlap

Largest trans-Neptunian objects

#	Name	Dynamical class	Radius (km)	Albedo	a (AU)	e	i (deg)	P_{rot} (hr)
134340	Pluto	RKBO	1185 ± 10	0.5	39.482	0.249	17.14	6.4
136199	Eris	SDO	1163 ± 12	0.69	67.728	0.44	43.97	
136472	Makemake	RKBO	750 ± 150	0.78	45.678	0.16	29.00	
136108	Haumea	SDO	675 ± 125	0.84	43.329	0.19	28.21	3.92
	Charon	Moon	606 ± 1.5	0.375	39.482	0.249	17.14	6.4
90377	Sedna	IOC	<800	>0.16	489.6	0.84	11.93	10.27
84522	2002 TC_{302}	SDO	575 ± 170	0.03	45.678	0.16	29.00	
90482	Orcus	RKBO	450 ± 40	0.28	39.363	0.22	20.59	
50000	Quaoar	CKBO	422 ± 100	0.20	43.572	0.04	7.98	17.68
55565	2002 AW_{197}	SDO	367 ± 160	0.12	47.349	0.13	24.39	

- CKBO: Classical KBO
- RKBO: Resonant KBO
- SDO: Scattered disc object
- IOC: Inner Oort Cloud

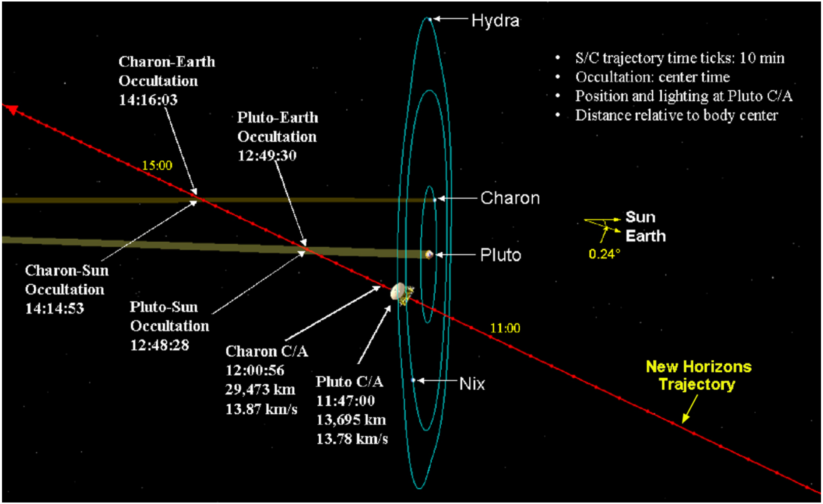
Relative sizes



- Orcus is about the same size as Ceres ($R = 450$ km)

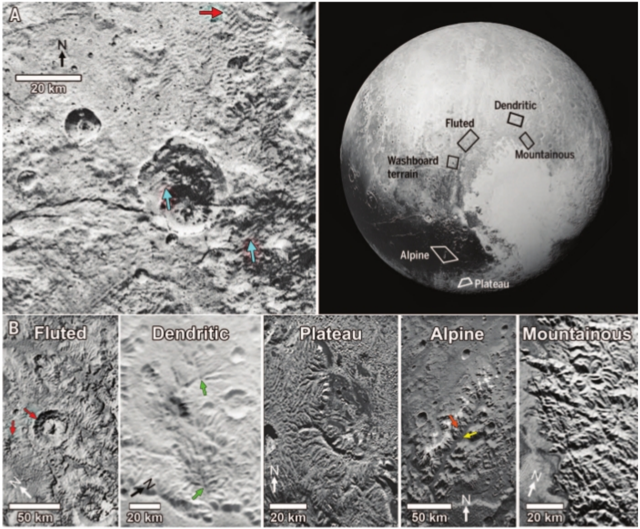
$\Rightarrow$ Largest trans-Neptunian objects are much larger than largest asteroids

New Horizon's flyby of Pluto



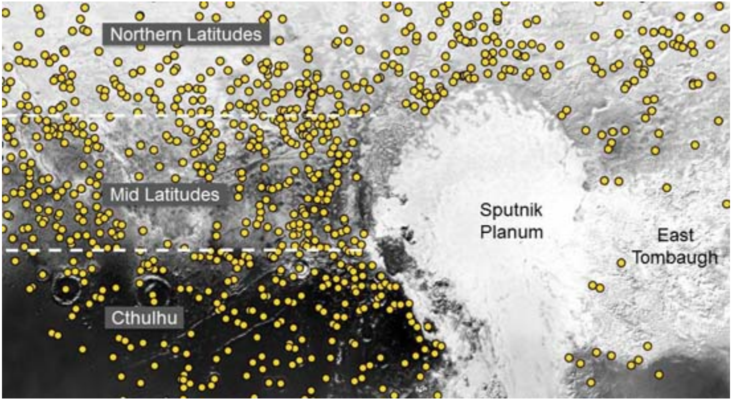
- S/C trajectory time ticks: 10 min
- Occultation: center time
- Position and lighting at Pluto C/A
- Distance relative to body center

Surface of Pluto



- Huge varieties of terrains on Pluto's surface

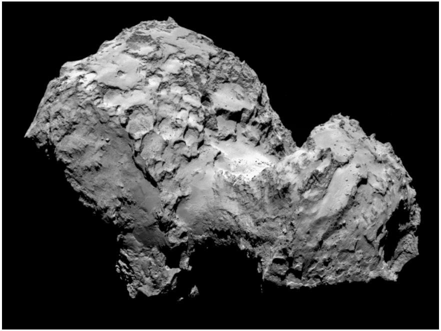
Craters on Pluto



- No cratering suggest a young surface, less than 10 Myr

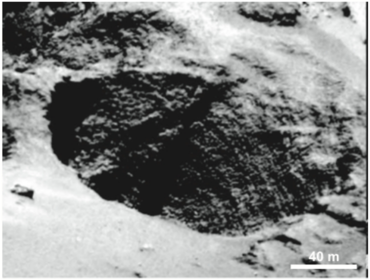
⇒ Impact basin filled with volatile ices (Nitrogen, CO)?

67P/Churyumov-Gerasimenko

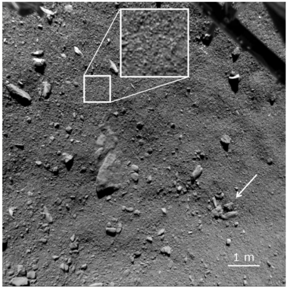


- Comets are icy objects from the Kuiper belt or the Oort cloud which enter the inner Solar System
- Some comets like Halley return periodically
- European Rosetta spacecraft orbits comet 67P

Goosebumps on 67P



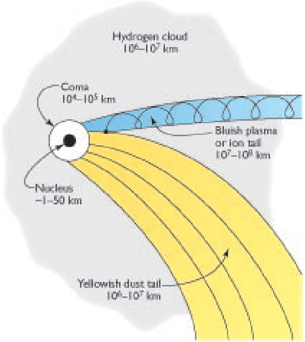
(*Sierks et al.*, 2015)

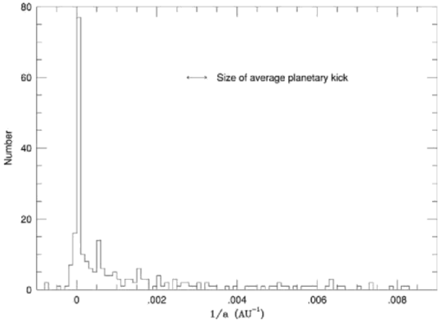


(*Mottola et al.*, 2015)

- The Rosetta mission arrived at the comet 67P/Churyumov-Gerasimenko in 2014
- Orbiter will follow 67P beyond perihelion
- Structures in deep pits resemble goosebumps (*Sierks et al., 2015*)
- Could be the primordial pebbles from the solar protoplanetary disc
- But meter-sized pebbles hard to explain in light of radial drift
- Philae's first landing site shows characteristic particle scale of cm in smooth terrains (*Mottola et al. 2015*)

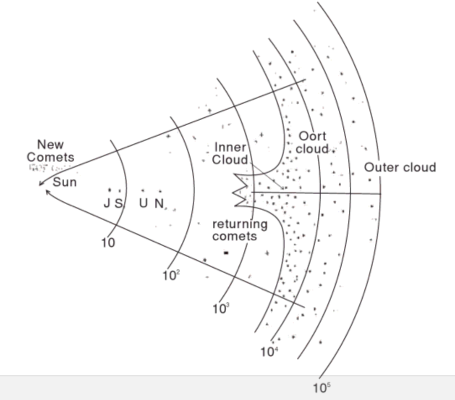
Comets





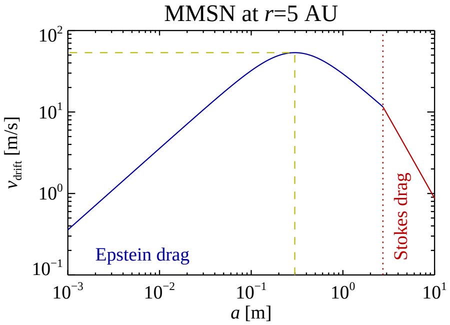
- Comets in the inner solar system are typically 1–10 km in size and consist mainly of water ice, refractory particles and organic compounds
- Comets come in two flavours: short-period comets and long-period comets
- Short-period comets are prograde and originate from the scattered disc
- Long-period comets come from random directions
- Hypothesized Oort cloud is source of long-period comets

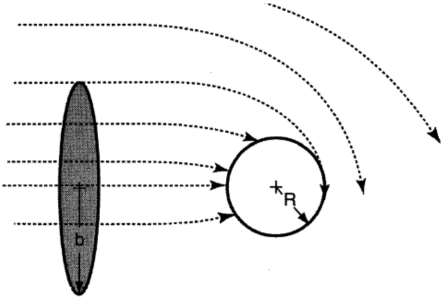
Scattered disc, Kuiper belt, Oort cloud



- Scattered disc contains approximately one Earth mass
- These objects have likely been scattered outwards by Neptune
- Classical Kuiper belt is far less massive, probably 0.01 Earth masses

From planetesimals to protoplanets



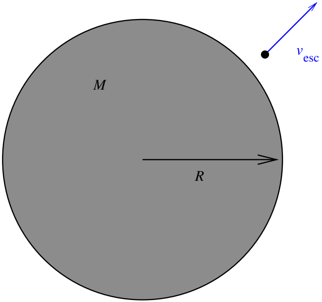


When particles reach planetesimal (>km) sizes

- they are no longer affected by gas drag, so orbits are maintained
- they exert a significant gravity on each other which leads to fast growth

⇒ **Next growth stage:** from planetesimals to protoplanets

Accretion of planetesimals



- Escape speed:

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}}$$

- Use mass $M = (4/3)\pi\rho_\bullet R^3$ for constant density sphere:

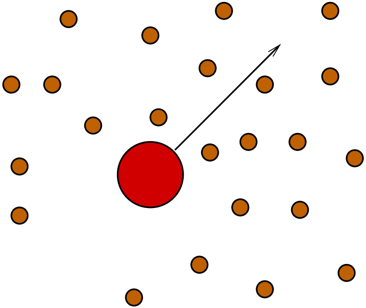
$$v_{\mathrm{esc}} = 0.15\,\frac{\mathrm{km}}{\mathrm{s}}\left(\frac{R}{100\,\mathrm{km}}\right)\left(\frac{\rho_\bullet}{4\,\mathrm{g\,cm^{-3}}}\right)^{1/2}$$

- Planetesimals are bound by gravity rather than material strength

⇒ Planetesimals can survive much higher collision speeds than dust particles

⇒ Large planetesimals continue to grow by colliding with smaller planetesimals

Mass growth rate



- Consider planetesimal with radius R and cross section πR^2
- Relative speed v relative to ocean of smaller planetesimals
- Mass density of planetesimal swarm in the neighbourhood $\rho_{\rm s}$
- Mass accretion rate (cross section × mass flux)

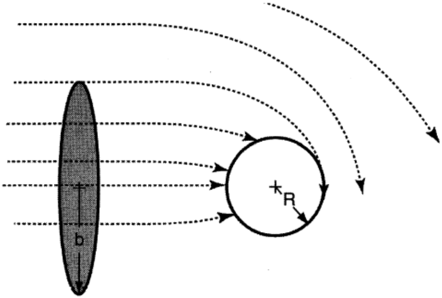
$$\frac{{\rm d}M}{{\rm d}t} = \pi R^2 v \rho_{\rm s} \mathcal{F}_{\rm g}$$

- Gravitational enhancement factor $\mathcal{F}_{\rm g}$ can be $\gg 1$

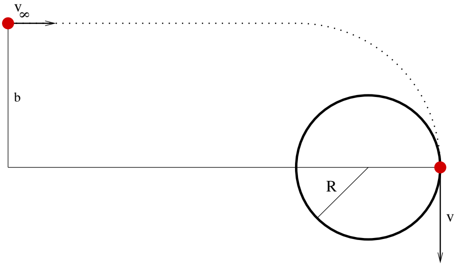
Gravitational cross section

- Particles arriving within impact parameter *b* are deflected by the planetesimal's gravity and accreted

⇒ Gravitating particles have collisional cross section much larger than their physical cross section



Gravitational cross section



- The most distant particle to hit the planetesimal arrives parallel to the surface with velocity v
- We can use conservation of energy and angular momentum to find b

$$\begin{aligned} \frac{1}{2}v_\infty^2 &= \frac{1}{2}v^2 - \frac{GM}{R} \\ bv_\infty &= vR \end{aligned}$$

- The solution is

$$\frac{b^2}{R^2} = \frac{v^2}{v_\infty^2} = 1 + \frac{2GM}{Rv_\infty^2} = 1 + \frac{v_{\rm esc}^2}{v_\infty^2}$$

Safronov number

Gravitational cross section

$$\sigma = \pi b^2 = \pi R^2 \left(1 + \frac{v_{\mathrm{esc}}^2}{v_\infty^2}\right) = \pi R^2 (1 + 2\theta_{\mathrm{S}})$$

$$\theta_{\mathrm{S}} = \frac{1}{2}\frac{v_{\mathrm{esc}}^2}{v_\infty^2} = \text{Safronov number}$$

- Mass accretion rate (cross section × mass flux)

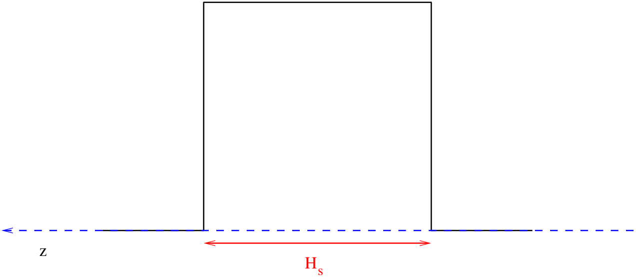
$$\frac{\mathrm{d}M}{\mathrm{d}t} = \pi R^2 v \rho_{\mathrm{s}} (1 + 2\theta_{\mathrm{S}})$$

- Use $M = (4/3)\pi R^3 \rho_\bullet$ to get $\dot{R}$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{v}{4}\frac{\rho_{\mathrm{s}}}{\rho_\bullet}(1 + 2\theta_{\mathrm{S}})$$

- Here $\rho_\bullet \approx 4\,\mathrm{g\,cm^{-3}}$ is the material density of rock
- Radius grows *linearly* in time
- But what is ρ_{s} of the planetesimal swarm?

Scale height of planetesimal swarm



- We know the planetesimal swarm's column density Σ_{s} from MMSN or other nebula model

- The swarm's space density is $\rho_{\mathrm{s}} \sim \Sigma_{\mathrm{s}}/H_{\mathrm{s}}$

- The *swarm scale height* is connected to the velocity dispersion through $H_{\mathrm{s}} \sim v/\Omega$

Growth rate of largest planetesimals

- Radius grows linearly with time

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{v}{4}\frac{\rho_\mathrm{s}}{\rho_\bullet}(1+2\theta_\mathrm{S})$$

- A detailed analysis of the planetesimal swarm density ρ_s gives

$$\Sigma_\mathrm{s} = \sqrt{\frac{\pi}{3}}\frac{\rho_\mathrm{s} v}{\Omega}$$

- The radius thus grows as

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sqrt{\frac{3}{\pi}}\frac{\Sigma_\mathrm{s}\Omega}{4\rho_\bullet}(1+2\theta_\mathrm{S})$$

- Using MMSN column densities of rock and ice yields

$$\frac{\mathrm{d}R}{\mathrm{d}t} \approx 2.7\,\mathrm{cm\,yr}^{-1}\left(\frac{r}{\mathrm{AU}}\right)^{-3}\left(\frac{\rho_\bullet}{4\,\mathrm{g\,cm}^{-3}}\right)^{-1}(1+2\theta_\mathrm{s}) \quad \mathrm{for} \quad 0.27 < r < 2.7$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} \approx 11.6\,\mathrm{cm\,yr}^{-1}\left(\frac{r}{\mathrm{AU}}\right)^{-3}\left(\frac{\rho_\bullet}{4\,\mathrm{g\,cm}^{-3}}\right)^{-1}(1+2\theta_\mathrm{s}) \quad \mathrm{for} \quad 2.7 < r < 36$$

Run-away accretion

- Mass growth rate

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \pi R^2 v \rho_\mathrm{s} \left(1 + \frac{2GM}{Rv^2}\right) = \pi R^2 v \rho_\mathrm{s} \left(1 + \frac{(8\pi/3)\rho_\bullet G R^2}{v^2}\right)$$

- Mass growth rate without and with gravitational focusing

$$\begin{aligned} \dot{M} \propto R^2 \propto M^{2/3} \quad &\mathrm{for} \quad v \gg v_\mathrm{esc} \\ \dot{M} \propto R^4 \propto M^{4/3} \quad &\mathrm{for} \quad v \ll v_\mathrm{esc} \end{aligned}$$

- The time-scale for mass doubling is $M/\dot{M}$

$$\begin{aligned} t_\mathrm{growth} \propto M^{+1/3} \quad &\mathrm{for} \quad v \gg v_\mathrm{esc} \\ t_\mathrm{growth} \propto M^{-1/3} \quad &\mathrm{for} \quad v \ll v_\mathrm{esc} \end{aligned}$$

- No gravitational focusing: small bodies grow faster than large bodies
- With gravitational focusing: large bodies grow faster than smaller bodies
  $\Rightarrow$ **run-away accretion** of a few large bodies

Formation time-scales

$$\frac{\mathrm{d}R}{\mathrm{d}t} \approx 2.7\,\mathrm{cm\,yr^{-1}}\left(\frac{r}{\mathrm{AU}}\right)^{-3}\left(\frac{\rho_\bullet}{4\,\mathrm{g\,cm^{-3}}}\right)^{-1}(1+2\theta_\mathrm{s}) \quad \text{for} \quad 0.27 < r < 2.7$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} \approx 11.6\,\mathrm{cm\,yr^{-1}}\left(\frac{r}{\mathrm{AU}}\right)^{-3}\left(\frac{\rho_\bullet}{4\,\mathrm{g\,cm^{-3}}}\right)^{-1}(1+2\theta_\mathrm{s}) \quad \text{for} \quad 2.7 < r < 36$$

- Time-scale to build Earth at 1 AU:

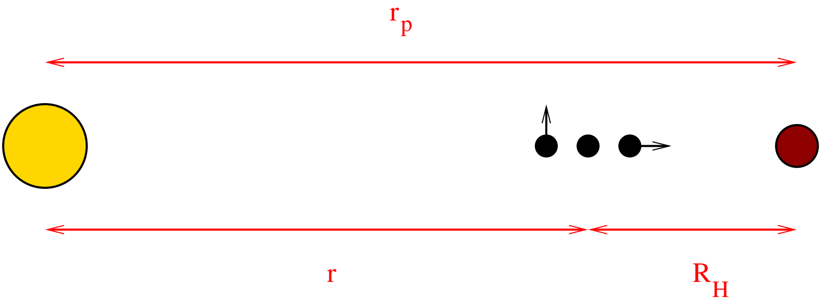
$$t_\oplus \approx 56\,\mathrm{Myr}\left(\frac{r}{\mathrm{AU}}\right)^3(1+2\theta_\mathrm{S})^{-1}$$

- Time-scale to build 10-Earth-mass core at 5 AU:

$$t_\mathrm{core} \approx 3500\,\mathrm{Myr}\left(\frac{r}{5\,\mathrm{AU}}\right)^3(1+2\theta_\mathrm{S})^{-1}$$

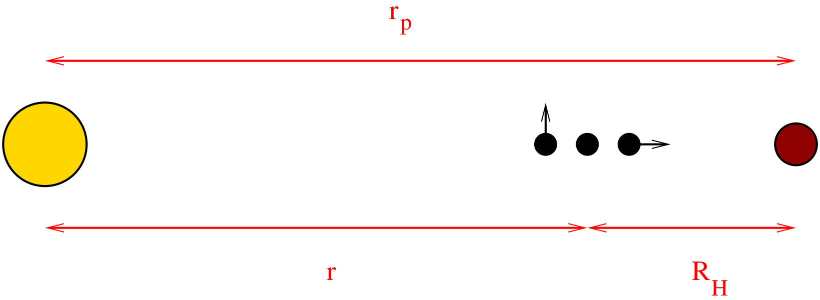
- More about formation of gas giant cores in the next lecture

Gravitational influence of planetesimals



- Planet acts as effective gravity reduction on test particle
- Three possibilities:
  1. $\Omega_{\rm t} > \Omega_{\rm p}$: test particle is slowed down by embryo but still moves away by differential rotation
  2. $\Omega_{\rm t} = \Omega_{\rm p}$: test particle acquires same angular frequency as the embryo
  3. $\Omega_{\rm t} < \Omega_{\rm p}$: embryo's gravity dominates over tidal force from the star and the test particle moves towards the embryo

Hill sphere

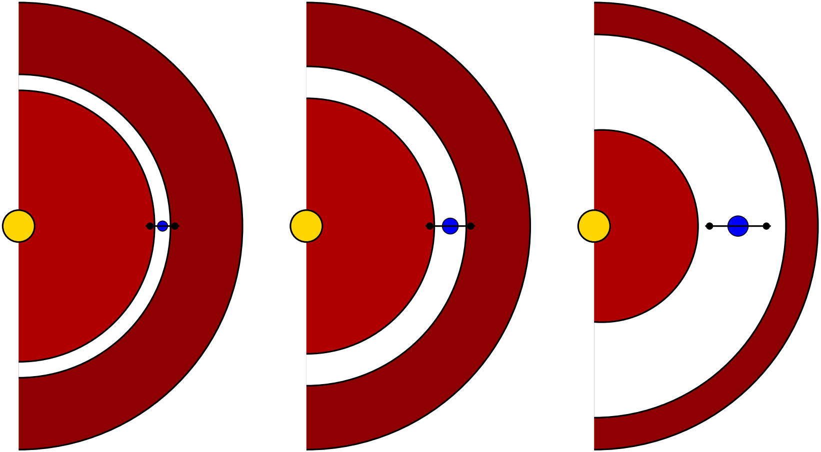


Planet's region of influence:

$$R_{\mathrm{H}}^3 = \frac{GM_{\mathrm{p}}}{3\Omega_{\mathrm{p}}^2} = \frac{M_{\mathrm{p}}}{3M_\star} r_{\mathrm{p}}^3$$

- R_{H} is the *Hill sphere*, named after George William Hill (1838 - 1914)
- A planetesimal or protoplanet can only accrete particles present inside its Hill sphere
- Particles further away move away from the planet because of differential rotation

Isolation mass



- Planetesimals can only accrete mass from within ≈ 4 Hill radii from their orbits ⇒ **reach isolation mass**

Isolation mass

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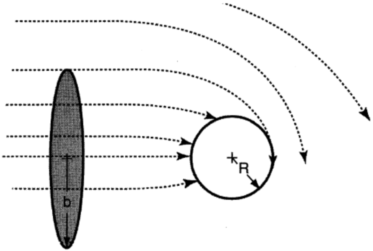
$$M_{\rm p} \approx 2\pi r (2\Delta r) \Sigma_{\rm s}$$

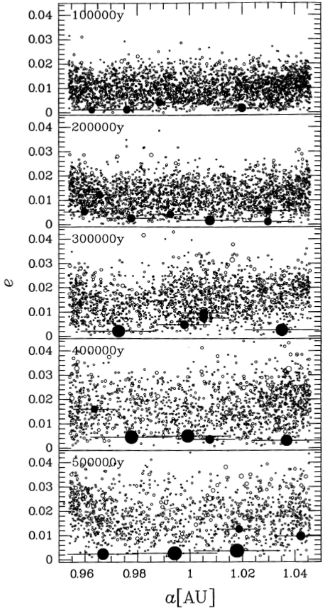
- Use $\Delta r = 4R_{\rm H}$ to get isolation mass in MMSN

$$\begin{aligned} M_{\rm iso} &\approx 3.8 M_{☾} \left(\frac{r}{\rm AU}\right)^{3/4} \left(\frac{M_\star}{M_\odot}\right)^{-1/2} && \text{for} \quad 0.27 < r < 2.7 \\ M_{\rm iso} &\approx 34.0 M_{☾} \left(\frac{r}{\rm AU}\right)^{3/4} \left(\frac{M_\star}{M_\odot}\right)^{-1/2} && \text{for} \quad 2.7 < r < 36 \end{aligned}$$

⇒ Protoplanets (or planetary embryos) in the terrestrial planet region have masses similar to Earth's moon

End of run-away accretion





- Particles in planetesimal swarm suffer close encounters with embryos and their speeds are excited towards the escape speed of the largest body
  ⇒ run-away accretion terminates

⇒ **Oligarchic growth** *(Kokubo & Ida 1998)*

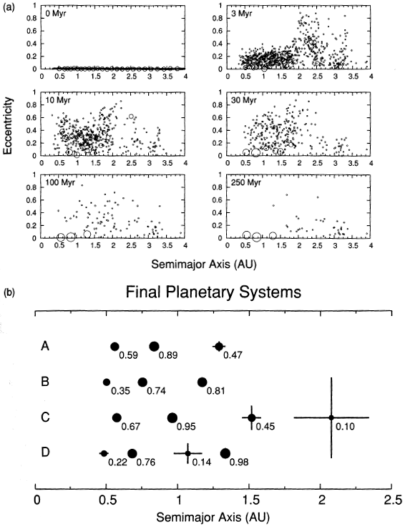
From embryos to terrestrial planets

- Moon-mass embryos are isolated by several Hill radii

- Perturb each other gravitationally until orbits cross

⇒ **Giant impact stage**

- Form 2–8 terrestrial planets in 10^8 years



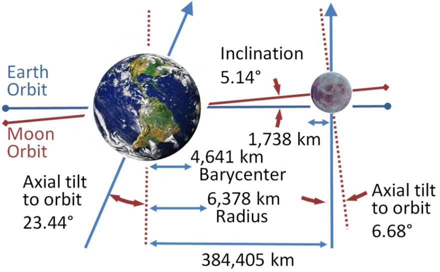
(O'Brien et al. 2006)

Some outstanding problems for terrestrial planet formation

- Based on rather arbitrary assumption that all dust turns to planetesimals at the same time
- Giant impact stage tends to form too few planets and too eccentric
- Planets get random rotation, but both Earth, Mars and the largest asteroids are prograde rotators
- Main problem: actually gas can *not* be ignored since there may still be many small bodies
- Future: include gas and hydrodynamics, couple with dust growth and planetesimal formation

Moon-Earth system

- $M_{\leftmoon} = 7.3477 \times 10^{22}\,\mathrm{kg} \approx 0.0123 M_{\oplus}$
- $r_{\leftmoon} = 384{,}399\,\mathrm{km} \approx 60 R_{\oplus}$



Earth Orbit

Moon Orbit

Axial tilt to orbit 23.44°

Inclination 5.14°

1,738 km

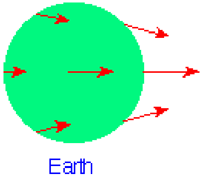
4,641 km Barycenter

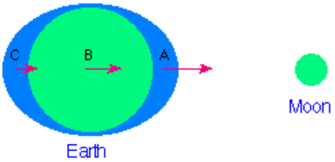
6,378 km Radius

Axial tilt to orbit 6.68°

384,405 km

Tides



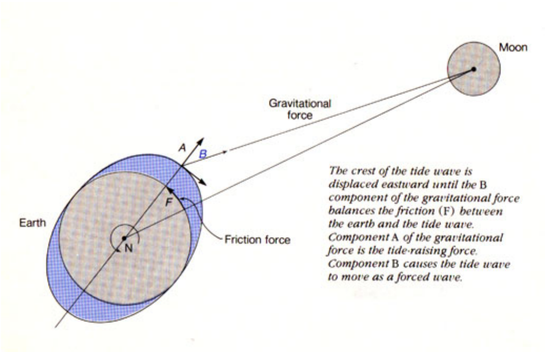


- The distance difference from the Moon to the near and the far side of the Earth leads to a differential gravity pull (*tidal force*)

- Rock is difficult to deform by tides, but the Earth's oceans react to the lunar tide and form a *tidal bulge* ($\sim$ 50 cm)

- The Moon also feels the tidal pull of the Earth, causing moonquakes (these occur because the Moon's orbit is eccentric, but the exact reason is not certain)

Tidal friction



The crest of the tide wave is displaced eastward until the B *component of the gravitational force balances the friction* (F) *between the earth and the tide wave. Component* A *of the gravitational force is the tide-raising force. Component* B *causes the tide wave to move as a forced wave.*

- The Earth spins around its axis in 24 hours
- The Moon orbits Earth in 27.3 days

⇒ Friction with Earth moves tidal bulge to lead the Moon's orbit

⇒ Earth's rotation slowed down by gravitational torque on tidal bulge

⇒ Gravitational torque between deformed Earth and Moon gives the Moon angular momentum so that its orbit expands (by ≈4 cm per year)

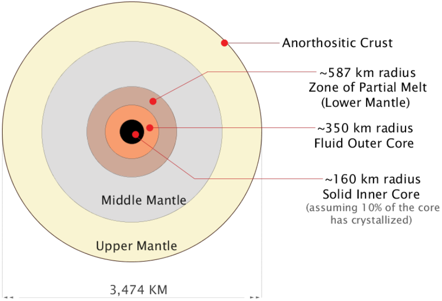
Angular momentum

- Moon formed much closer to Earth, at a distance of ∼50,000 km

- Angular momentum conservation gives an original spin period of the Earth of only 6 hours

- Tides on Earth were huge, more than 50 meters

Structure of the Moon



- The Moon's mean density is very low, with uncompressed density $\rho = 3.3\,\mathrm{g\,cm^{-3}}$ [Earth's uncompressed density: $\rho = 4.4\,\mathrm{g\,cm^{-3}}$]
- The Moon is highly differentiated – with a dense core, a mantle, and a crust – but must be lacking iron
- Surface consists of very-low-density Anorthosite (feldspar with minimal mafic component)

$\Rightarrow$ Moon was entirely molten when born (*magma ocean*) and differentiated by fractional crystallisation

Moon formation

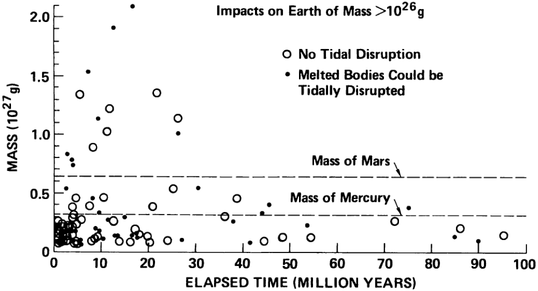
1. The Moon is depleted in iron
2. The Moon formed close to the Earth
3. The Moon was very hot when it formed

⇒ The Moon formed from ejecta from a giant impact between Earth and a Mars-sized protoplanet

Giant impact stage

The three stages of terrestrial planet formation:

1. Dust to planetesimals (van der Waals forces and gravitational instability)
2. Planetesimals to protoplanets (run-away accretion)
3. Protoplanets to planets (giant impacts)



(Wetherill 1985)

⇒ Giant impacts are a completely natural by-product of planet formation

Angular momentum in Moon-forming collision

- The current orbital angular momentum of the Moon:

$$L_{☾} = M_{☾} \times r_{☾} \times v_{☾} \approx 3 \times 10^{34}\,\mathrm{kg\,m^2\,s^{-1}}$$

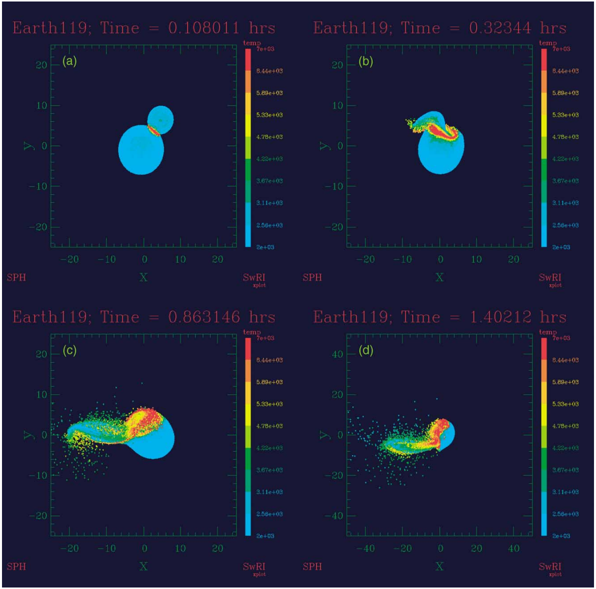
- An impact with body of mass M, with impact parameter b and velocity v, has angular momentum

$$\begin{aligned} L_{\mathrm{imp}} &= M \times d \times v \\ &\approx 4.3 \times 10^{35}\,\mathrm{kg\,m^2\,s^{-1}} \left(\frac{M}{M_\oplus}\right) \left(\frac{b}{R_\oplus}\right) \left(\frac{v}{11.2\,\mathrm{km/s}}\right) \end{aligned}$$

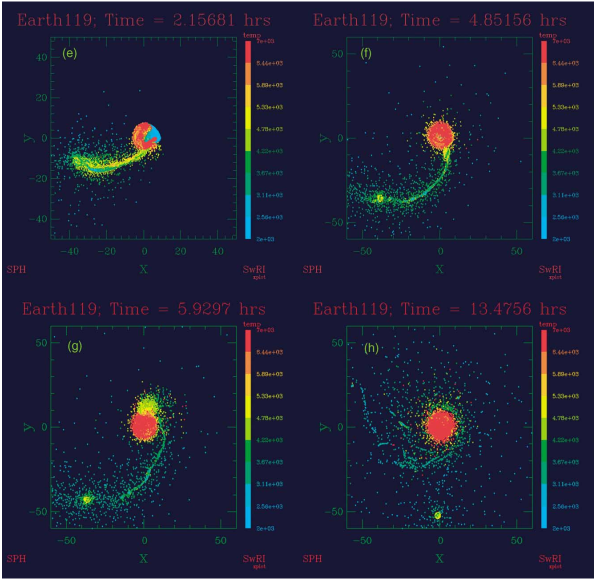
⇒ Collision with $\sim 0.1\ M_\oplus$ (approximately Mars-mass) body can explain angular momentum (**Theia**)

Artist's impression of Theia

Simulations by Canup (2004)



Simulations by Canup (2004)



Summary

- Dust particles can collide and grow to pebbles, but growth is limited by the fragmentation and radial drift barrier
- Pebbles can concentrate in pressure bumps and via the streaming instability, so that a collapsing pebble cloud forms planetesimals
- Planetesimals can grow to embryos, that reach reach isolation mass ($\sim M_{☾}$ in terrestrial planet formation region)
- Embryos perturb each other's orbits over 10 to 100 million years
- Final assembly of terrestrial planets through giant impact phase
- The formation of the Moon after proto-Earth collides with a Mars-sized protoplanet is a natural consequence of the giant impact stage